

On the Reflection Type Decomposition of the Adjoint Reduced Phase Space of a Compact Semisimple Lie group

M. Hofmann[†], G. Rudolph^{*} and M. Schmidt^{*}

[†] Naturwissenschaftlich-Technische Fakultät, Universität Siegen
Walter-Flex-Str. 3, 57068 Siegen, Germany

^{*} Institut für Theoretische Physik, Universität Leipzig
Augustusplatz 10/11, 04109 Leipzig, Germany

February 26, 2013

Abstract

We consider a system with symmetries whose configuration space is a compact Lie group, acted upon by inner automorphisms. The classical reduced phase space of this system decomposes into connected components of orbit type subsets. To investigate hypothetical quantum effects of this decomposition one has to construct the associated costratification of the Hilbert space of the quantum system in the sense of Huebschmann. In the present paper, instead of the decomposition by orbit types, we consider the related decomposition by reflection types (conjugacy classes of reflection subgroups). These two decompositions turn out to coincide e.g. for the classical groups $SU(n)$ and $Sp(n)$. We derive defining relations for reflection type subsets in terms of irreducible characters and discuss how to obtain from that the corresponding costratification of the Hilbert space of the system. To illustrate the method, we give explicit results for some low rank classical groups.

1 Introduction

This paper is part of a program which aims at developing a non-perturbative approach to the quantum theory of gauge fields in the Hamiltonian framework with special emphasis on the role of non-generic gauge orbit strata. The starting point is a finite-dimensional lattice approximation of the theory.¹ In this context, one obtains a finite-dimensional Hamiltonian system with symmetries and one can perform Marsden-Weinstein reduction. The configuration space and, consequently, the phase space of this system both have a stratified structure, which was studied in [6, 7, 13]. On quantum level, we have investigated the structure of the observable algebra [23, 24, 25, 30], including the study of the superselection structure, and we have obtained some insight into the role of the non-generic gauge orbit strata [19, 22, 29]. The latter is based on an idea of Huebschmann and will be explained in Section 2 in detail. In short, it may be described as follows. One views the states of the unreduced system as elements of the Hilbert space \mathcal{H} of square integrable holomorphic functions, implements the symmetry reduction on quantum level and, for a given classical gauge orbit stratum, one considers the subspace of \mathcal{H} consisting of functions vanishing on this stratum. Next, one takes its orthogonal complement in \mathcal{H} . This is, by definition, the Hilbert space of physical states associated with the stratum under consideration. Performing this procedure for every element of the stratification, one obtains a partially ordered family of closed subspaces of \mathcal{H} called the costratification associated with the classical gauge orbit stratification. In [22] we studied this procedure for a toy model with gauge group $SU(2)$.

The aim of the present paper is to make a step towards extending the results of [22] to an arbitrary compact connected semisimple Lie group G . Let us explain the setting. We consider a physical system with configuration space G , acted upon by itself by inner automorphisms. As usual, we will refer to this action as the adjoint action of G . This models a lattice gauge theory with structure group G on a single plaquette in a tree gauge, together with the residual gauge transformations. The phase space is given by the cotangent bundle T^*G , acted upon by the lifted action. This action is symplectic and possesses a natural momentum mapping $\mu : T^*G \rightarrow \mathfrak{g}^*$, where \mathfrak{g} denotes the Lie algebra of G [1]. Using an invariant scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{g} and left translation on G , we can define a global trivialization of T^*G by

$$G \times \mathfrak{g} \rightarrow T^*G, \quad (a, A) \mapsto \langle A_a, \cdot \rangle, \quad (1)$$

where A has to be interpreted as a left-invariant vector field and A_a means the value of A at a . In this trivialization, the lifted action reads

$$g \cdot (a, A) = (gag^{-1}, \text{Ad}(g)A),$$

where $a, g \in G$ and $A \in \mathfrak{g}$, and the momentum mapping is given by

$$\mu(a, A) = \text{Ad}(a)A - A. \quad (2)$$

¹Of course, ultimately, one is interested in constructing the continuum limit, see [15] for a first step in this direction.

The reduced configuration space \mathcal{X} is given by the adjoint quotient, that is, by the set of orbits of the adjoint action, endowed with the quotient topology. Let \mathcal{T} denote the set of orbit types of the adjoint action. For $\tau \in \mathcal{T}$, let \mathcal{X}_τ denote the subset of \mathcal{X} consisting of the orbits of type τ . A general theorem on proper Lie group actions implies that the family $\{\mathcal{X}_{\tau,i}\}$, where $\tau \in \mathcal{T}$ and i labels the connected components of \mathcal{X}_τ , forms a stratification of \mathcal{X} [28]. This means, in particular, that the connected components $\mathcal{X}_{\tau,i}$ are smooth manifold and that the condition of the frontier holds: if \mathcal{X}_{τ_2,i_2} intersects the closure $\overline{\mathcal{X}_{\tau_1,i_1}}$ of \mathcal{X}_{τ_1,i_1} in \mathcal{X} , then $\mathcal{X}_{\tau_2,i_2} \subset \overline{\mathcal{X}_{\tau_1,i_1}}$. The condition of the frontier gives rise to a natural partial ordering on the family $\{\mathcal{X}_{\tau,i}\}$: $\mathcal{X}_{\tau_1,i_1} \leq \mathcal{X}_{\tau_2,i_2}$ iff $\mathcal{X}_{\tau_2,i_2} \subset \overline{\mathcal{X}_{\tau_1,i_1}}$. One can show that this partial ordering is compatible with the natural partial ordering of orbit types by inclusion modulo conjugacy. That is,

$$\mathcal{X}_{\tau_1,i_1} \leq \mathcal{X}_{\tau_2,i_2} \quad \Rightarrow \quad \tau_1 \leq \tau_2, \quad (3)$$

where $\tau_1 \leq \tau_2$ if there exist representatives H_1 of τ_1 and H_2 of τ_2 such that $H_1 \subseteq H_2$. The reduced phase space \mathcal{P} is obtained from T^*G by singular symplectic reduction at zero level. That is, \mathcal{P} is the set of orbits of the lifted action of G on the invariant subset $\mu^{-1}(0) \subseteq T^*G$, endowed with the quotient topology induced from the relative topology on this subset. In lattice gauge theory, the condition $\mu = 0$ corresponds to the Gauß law constraint. As is true in general for the lift of a Lie group action and the associated momentum mapping, the action on $\mu^{-1}(0)$ has the same orbit types as the original action, i.e., \mathcal{T} . By the procedure of singular symplectic reduction, the connected components $\mathcal{P}_{\tau,i}$ of the orbit type subsets \mathcal{P}_τ of \mathcal{P} , $\tau \in \mathcal{T}$, are endowed with natural symplectic manifold structures. Moreover, they provide a stratification of \mathcal{P} [34, 27]. In particular, the condition of the frontier holds for the subsets $\mathcal{P}_{\tau,i}$ as well and the induced partial ordering on the family $\{\mathcal{P}_{\tau,i}\}$ is compatible with the natural partial ordering of \mathcal{T} in the sense of (3). The bundle projection $\pi : T^*G \rightarrow G$ induces a mapping $\hat{\pi} : \mathcal{P} \rightarrow \mathcal{X}$. This mapping is surjective, because μ is linear on the fibres of T^*G and hence $\mu^{-1}(0)$ contains the zero section of T^*G . Note that $\hat{\pi}$ need not preserve the orbit type, but, as π is equivariant, one has at least

$$\hat{\pi}(\mathcal{P}_\tau) \subseteq \bigcup_{\tau' \geq \tau} \mathcal{X}_{\tau'}.$$

2 Stratified quantum theory on an adjoint quotient

To investigate hypothetic quantum effects of the orbit type decomposition of the reduced phase space, one has to implement this decomposition on quantum level. Following Huebschmann [19, 22], this can be done by means of a family of closed subspaces of the Hilbert space as follows. First, we pass to the quantum theory of the reduced system by means of geometric (Kähler) quantization on T^*G and subsequent reduction. Let $\mathfrak{g}^\mathbb{C}$ denote the complexification of \mathfrak{g} and let $G^\mathbb{C}$ denote the complexification of G . The inverse of the polar decomposition on $G^\mathbb{C}$ yields a diffeomorphism

$$G \times \mathfrak{g} \rightarrow G^\mathbb{C}, \quad (a, A) \mapsto a \exp(iA),$$

where \exp denotes the exponential mapping of $G^{\mathbb{C}}$. Composition of this diffeomorphism with the global trivialization (1) yields a diffeomorphism $T^*G \cong G^{\mathbb{C}}$ which intertwines the lifted action of G on T^*G with the action of the subgroup $G \subset G^{\mathbb{C}}$ on $G^{\mathbb{C}}$ by inner automorphisms. Via this diffeomorphism, the symplectic structure of T^*G and the complex structure of $G^{\mathbb{C}}$ combine to a Kähler structure. Half-form Kähler quantization on $G^{\mathbb{C}}$ yields the Hilbert space $HL^2(G^{\mathbb{C}}, d\nu)$ of holomorphic functions on $G^{\mathbb{C}}$ which are square-integrable with respect to a certain measure, $d\nu$, containing the Liouville measure on T^*G , the Kähler potential on $G^{\mathbb{C}}$ and the half-form correction [18]. Reduction then yields the closed subspace $\mathcal{H} = HL^2(G^{\mathbb{C}}, d\nu)^G$ of G -invariants as the Hilbert space of the reduced system. For convenience, we realize \mathcal{H} in two different ways. First, the holomorphic Peter-Weyl theorem [21] implies that \mathcal{H} is spanned by the irreducible characters $\chi_{\lambda}^{\mathbb{C}}$ of $G^{\mathbb{C}}$. Here, $\lambda \in \Lambda_+$, where Λ_+ denotes the set of the highest weights of the finite-dimensional irreducible complex representations of $G^{\mathbb{C}}$, relative to some chosen dominant Weyl chamber in some chosen Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. The system $\{\chi_{\lambda}^{\mathbb{C}} : \lambda \in \Lambda_+\}$ is orthogonal but not normalized. According to [18] or [21, Lemma 3.3], the norms are

$$N_{\lambda} := \|\chi_{\lambda}^{\mathbb{C}}\| = (\pi\hbar)^{\dim G/4} e^{\hbar\|\lambda+\delta\|^2/2},$$

where \hbar is the Planck constant, the norm $\|\cdot\|$ on $(\mathfrak{g}^{\mathbb{C}})^*$ is defined by a chosen invariant scalar product on \mathfrak{g} and δ denotes half the sum of the roots which are positive in the sense of the chosen dominant Weyl chamber,

$$\delta = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha. \quad (4)$$

Both \hbar and the scaling of the invariant scalar product appear as parameters in the measure $d\nu$. Thus, using the orthonormal basis of normed complex characters

$$\tilde{\chi}_{\lambda}^{\mathbb{C}} := \frac{\chi_{\lambda}^{\mathbb{C}}}{N_{\lambda}}, \quad \lambda \in \Lambda_+,$$

and endowing Λ_+ with the lexicographic ordering defined by the system of simple roots associated with the chosen Weyl chamber, we may identify \mathcal{H} with the Hilbert space ℓ^2 over the index set Λ_+ . Second, the ordinary Peter-Weyl theorem for G implies that the irreducible characters χ_{λ} of G provide an orthonormal basis in the closed subspace $L^2(G, d\mu)^G$ of G -invariant functions in the Hilbert space $L^2(G, d\mu)$ of functions on G which are square-integrable with respect to the normalized Haar measure $d\mu$. Consequently, the assignment of χ_{λ} to $\chi_{\lambda}^{\mathbb{C}}/\|\chi_{\lambda}^{\mathbb{C}}\|$ for every $\lambda \in \Lambda_+$ yields a Hilbert space isomorphism

$$\mathcal{H} \cong L^2(G, d\mu)^G. \quad (5)$$

Remark 2.1. The Hilbert space isomorphism (5) coincides with the restriction to the G -invariants of the Segal-Bargmann transformation $L^2(G, d\mu) \rightarrow HL^2(G^{\mathbb{C}}, \nu)$. Originally, the Segal-Bargmann transformation for compact Lie groups was developed via heat kernel analysis on G and $G^{\mathbb{C}}$ in [17]. Alternatively, as a consequence of the holomorphic Peter-Weyl theorem cited above, it can be described in terms of representative functions on G and $G^{\mathbb{C}}$. This is the approach we have used here. \blacklozenge

Now, we can define the costratified Hilbert space structure associated with the decomposition of \mathcal{P} by the connected components $\mathcal{P}_{\tau,i}$ of the orbit type subsets of \mathcal{P} . Since the elements of \mathcal{H} are holomorphic functions on $G^{\mathbb{C}}$ (and not classes of functions as in the L^2 -case), and since they are G -invariant, they descend to continuous functions on \mathcal{P} . Thus, to every stratum $\mathcal{P}_{\tau,i}$ there corresponds a subspace $\mathcal{V}_{\tau,i}$ consisting of the functions in $HL^2(G^{\mathbb{C}}, d\nu)^G$ which vanish on $\mathcal{P}_{\tau,i}$. Then, the subspace $\mathcal{H}_{\tau,i}$ associated with the stratum $\mathcal{P}_{\tau,i}$ is defined to be the orthogonal complement of $\mathcal{V}_{\tau,i}$ in $HL^2(G^{\mathbb{C}}, d\nu)^G$. If $\mathcal{P}_{\tau_1,i_1} \leq \mathcal{P}_{\tau_2,i_2}$, then $\mathcal{P}_{\tau_2,i_2} \subset \overline{\mathcal{P}_{\tau_1,i_1}}$ and hence $\mathcal{V}_{\tau_1,i_1} \subseteq \mathcal{V}_{\tau_2,i_2}$. Then, $\mathcal{H}_{\tau_2,i_2} \subseteq \mathcal{H}_{\tau_1,i_1}$, so that to every pair $\mathcal{P}_{\tau_1,i_1}, \mathcal{P}_{\tau_2,i_2}$ satisfying $\mathcal{P}_{\tau_1,i_1} \leq \mathcal{P}_{\tau_2,i_2}$ there corresponds a bounded linear mapping $\Pi_{\tau_1,i_1;\tau_2,i_2} : \mathcal{H}_{\tau_1,i_1} \subseteq \mathcal{H}_{\tau_2,i_2}$, given by orthogonal projection. The families $\{\mathcal{H}_{\tau,i}\}$ and $\{\Pi_{\tau_1,i_1;\tau_2,i_2}\}$ constitute a costratified Hilbert space in the sense of Huebschmann.

The reader will have noticed that the construction of the subspaces $\mathcal{V}_{\tau,i}$ and $\mathcal{H}_{\tau,i}$ does not make use of the fact that the decomposition of \mathcal{P} under consideration is by orbit type connected components. It can be constructed for any decomposition fulfilling the condition of the frontier. In fact, in this paper, we will not consider the decomposition of \mathcal{P} by orbit type connected components but the decomposition by reflection types \mathbf{r} , to be defined below. The reason is that this decomposition can be conveniently described in terms of roots. In many cases, like $SU(n)$ and $Sp(n)$, the two decompositions coincide, see Remark 8.2. In general, the decomposition by reflection types is slightly coarser.

Thus, our goal is to find the subspaces $\mathcal{V}_{\mathbf{r}} \subseteq HL^2(G^{\mathbb{C}}, d\nu)$ of functions vanishing on the reflection type stratum $\mathcal{P}_{\mathbf{r}}$. We will also make some comments on how to get hands on their orthogonal complements $\mathcal{H}_{\mathbf{r}}$. A detailed study of the latter remains as a future task. To determine $\mathcal{V}_{\mathbf{r}}$, we use that if ν_1, \dots, ν_l are elements of $HL^2(G^{\mathbb{C}}, d\nu)$ such that $\mathcal{P}_{\mathbf{r}} = \bigcap_{i=1}^l \nu_i^{-1}(0)$, then $\mathcal{V}_{\mathbf{r}}$ is spanned by the functions

$$\nu_k \chi_{\lambda}^{\mathbb{C}}, \quad k = 1, \dots, l, \quad \lambda \in \Lambda_+.$$

Thus, for every \mathbf{r} we have to find a set of relations defining $\mathcal{P}_{\mathbf{r}}$ as a subset of \mathcal{P} .

3 Reduction to the Weyl group action

First, we show that, in the special situation under consideration, \mathcal{P} can be identified with the quotient of a Lie group action on a manifold. This is not true in general.

Let T be some chosen maximal toral subgroup of G , let \mathfrak{t} be the Lie algebra of T and let $W = N(T)/C(T)$ be the corresponding Weyl group. Via the adjoint representation, W acts on \mathfrak{t} . Let $T^{\mathbb{C}}$ be the image of $T \times \mathfrak{t}$ under the inverse of the polar decomposition, that is, $T^{\mathbb{C}} = T \exp(i\mathfrak{t})$. Since conjugation by elements of $C(T)$ leaves $T^{\mathbb{C}}$ pointwise invariant, the action of W on T extends to an action of W on $T^{\mathbb{C}}$. The polar decomposition is equivariant with respect to this action. Let $\mathfrak{t}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$ denote the Lie algebras of $T^{\mathbb{C}}$ and $G^{\mathbb{C}}$, respectively. Then, $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Moreover, \mathfrak{t} and \mathfrak{g} can be identified in an obvious way with subsets of $\mathfrak{t}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}}$, respectively.

Lemma 3.1. *The natural inclusion mapping $T^{\mathbb{C}} \rightarrow G^{\mathbb{C}}$ descends to a homeomorphism from the topological quotient $T^{\mathbb{C}}/W$ onto \mathcal{P} .*

In particular, in the present situation, the reduced phase space is an orbifold. The homeomorphism provided by the lemma induces a natural projection

$$\text{pr} : T^{\mathbb{C}} \rightarrow \mathcal{P}.$$

Proof. First, we show that for every $x \in G^{\mathbb{C}}$ satisfying $\mu(x) = 0$, there exists $g \in G$ such that $g x g^{-1} \in T^{\mathbb{C}}$. Let $(a, A) \in G \times \mathfrak{g}$ such that $x = a \exp(iA)$. According to (2), $\mu(a, A) = 0$ implies $\text{Ad}(a)A = A$. Then, $\exp(tA)$ belongs to the identity component of the centralizer of a in G for all $t \in \mathbb{R}$. By Cor. 3 of Thm. 2 in §IX.2.2 of [3], this component coincides with the union of all maximal tori in G which contain a . Hence, there exists a maximal torus containing a and $\exp(tA)$ for all t . Since all maximal tori are conjugate in G , there exists $g \in G$ such that both $g a g^{-1}$ and $g \exp(tA) g^{-1} = \exp(t \text{Ad}(g)A)$ belong to T , for all t . It follows that $\text{Ad}(g)A \in \mathfrak{t}$ and hence $g \cdot (a, A) \in T \times \mathfrak{t}$. Then, $g x g^{-1} \in T^{\mathbb{C}}$. Next, we show that two elements x, y of $T^{\mathbb{C}}$ are conjugate under G iff they are conjugate under W . For that purpose, it suffices to show that conjugacy under G implies conjugacy under the normalizer $N_G(T)$ of T in G . Thus, let $g \in G$ such that $g x g^{-1} = y$. Let $(a, A), (b, B) \in T \times \mathfrak{t}$ such that $x = a \exp(iA)$ and $y = b \exp(iB)$. Then, $g a g^{-1} = b$ and $\text{Ad}(g)A = B$, so that under conjugation by g , the subset $\{a, \exp tA\}$ of T is mapped to the subset $\{b, \exp tB\}$ of T for all t . By Cor. 7 of Thm. 2 in §IX.2.2 of [3], there exists $h \in G$ such that $ghT(gh)^{-1} = T$, $gha(gh)^{-1} = b$ and $gh \exp tA(gh)^{-1} = \exp tB$. The latter implies $\text{Ad}(gh)A = B$ and hence $ghx(gh)^{-1} = y$.

As a result of these considerations, the natural inclusion mapping $T \times \mathfrak{t} \rightarrow G \times \mathfrak{g}$ descends to a bijection from $(T \times \mathfrak{t})/W$ onto \mathcal{P} . The induced mapping is a homeomorphism: continuity follows from the fact that the preimage under the natural inclusion mapping $T \times \mathfrak{t} \rightarrow \mu^{-1}(0)$ of a subset of $\mu^{-1}(0)$ which is saturated with respect to the natural projection $\mu^{-1}(0) \rightarrow \mathcal{P}$ is saturated with respect to the natural projection $T \times \mathfrak{t} \rightarrow (T \times \mathfrak{t})/W$. Openness follows from the fact that for a Lie group action, both the action mapping and the natural projection to the orbit space are open mappings. \square

Let Σ denote the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$. Since $\mathfrak{g}^{\mathbb{C}}$ is semisimple, the Killing form k defines a vector space isomorphism between $\mathfrak{t}^{\mathbb{C}}$ and its dual. For $\alpha \in \Sigma$, let H_{α} denote the image of α under this isomorphism. That is,

$$k(H_{\alpha}, X) = \alpha(X)$$

for all $X \in \mathfrak{t}^{\mathbb{C}}$. The action of W on $T^{\mathbb{C}}$ induces a faithful representation of W on $\mathfrak{t}^{\mathbb{C}}$ which is orthogonal with respect to k and whose image is generated by the mappings

$$\sigma_{\alpha} : \mathfrak{t}^{\mathbb{C}} \rightarrow \mathfrak{t}^{\mathbb{C}}, \quad \sigma_{\alpha}(X) = X - 2 \frac{\alpha(X)}{k(\alpha, \alpha)} H_{\alpha}, \quad (6)$$

where $\alpha \in \Sigma$. Geometrically, σ_{α} corresponds to the k -orthogonal reflection about the hyperplane which is k -orthogonal to H_{α} . To σ_{α} there corresponds a unique element of W which will be denoted by the same symbol and which will be referred to as the reflection associated with α . Via the vector space isomorphism $\mathfrak{t}^{\mathbb{C}} \cong (\mathfrak{t}^{\mathbb{C}})^*$ induced by k , k defines

a symmetric bilinear form on $(\mathfrak{t}^\mathbb{C})^*$, denoted by the same symbol, and the representation of W on $\mathfrak{t}^\mathbb{C}$ induces a representation of W on $(\mathfrak{t}^\mathbb{C})^*$. Since the root system $\Sigma \subset (\mathfrak{t}^\mathbb{C})^*$ is invariant under this representation, the latter induces an action of W on Σ . For the reflections σ_α , (6) implies

$$\sigma_\alpha : \Sigma \rightarrow \Sigma, \quad \sigma_\alpha(\beta) = \beta - 2 \frac{k(\alpha, \beta)}{k(\alpha, \alpha)} \alpha. \quad (7)$$

Moreover, using that k is W -invariant, from (7) one can conclude that for any $w \in W$,

$$\sigma_{w(\alpha)} = w \circ \sigma_\alpha \circ w^{-1}. \quad (8)$$

The real vector space spanned by H_α , $\alpha \in \Sigma$, can be identified with the real subspace $\mathfrak{it} \subset \mathfrak{t}^\mathbb{C}$. Correspondingly, the real vector space spanned by Σ can be identified with the dual of it. Moreover, the lattice generated by the elements $2\pi i \frac{2H_\alpha}{k(\alpha, \alpha)}$ is contained in $\ker(\exp) \subset \mathfrak{t}$ and it coincides with the latter if G is simply connected [4, §V.2.16].²

4 Reflection types

A subgroup of W generated by reflections is commonly referred to as a reflection subgroup. The starting point of our analysis is the well-known fact that the stabilizer of an element X of $\mathfrak{t}^\mathbb{C}$ under W is a reflection subgroup. In fact, it is generated by the reflections σ_α for all $\alpha \in \Sigma$ with $\alpha(X) = 0$ [16, §4.1]. This may however not be true for elements of $T^\mathbb{C}$:

Example 4.1. Consider the projective unitary group $G = \text{PU}(3)$, given by the quotient of the special unitary group $\text{SU}(3)$ by its center. Let T be the subgroup of cosets consisting of diagonal matrices. The Weyl group W acts on these cosets by permuting the matrix entries of some representative. It is easy to see that the stabilizer of the coset of the matrix

$$\begin{bmatrix} 1 & 0 & \\ 0 & e^{i\frac{2\pi}{3}} & 0 \\ 0 & 0 & e^{i\frac{4\pi}{3}} \end{bmatrix}$$

consists of those elements of W which act as cyclic permutations. In particular, it does not contain any reflection. \blacklozenge

Since reflection subgroups can be conveniently described in terms of roots, instead of the stabilizers themselves, we will use their maximal reflection subgroups to construct a disjoint decomposition of $T^\mathbb{C}$ into W -invariant subsets. We will characterise these subsets in terms of roots and derive defining relations.

For a subgroup \tilde{W} of W , the maximal reflection subgroup of \tilde{W} is generated by the reflections associated with the elements of the subset

$$\Gamma_{\tilde{W}} := \{\alpha \in \Sigma : \sigma_\alpha \in \tilde{W}\} \quad (9)$$

²Note that in [4], the system R of real roots is used. One has $R = \frac{1}{2\pi i} \Sigma$.

of Σ . Let us describe the subsets of Σ arising this way. Following [9], by a root subsystem we mean a subset $\Gamma \subset \Sigma$ which is a root system in some linear subspace of $\text{span}_{\mathbb{R}}\Sigma$. For convenience, we include the case of the empty set. It is easy to see that Γ is a root subsystem iff it is invariant under the reflections associated with its elements, that is, iff $\sigma_{\alpha}(\beta) \in \Gamma$ for all $\alpha, \beta \in \Gamma$. Thus, as a consequence of Formula (8), $\Gamma_{\tilde{W}}$ is a root subsystem for every subgroup \tilde{W} of W . In fact, the assignment of $\Gamma_{\tilde{W}}$ to \tilde{W} induces a bijection between reflection subgroups of W and root subsystems of Σ .

Now, let $x \in T^{\mathbb{C}}$ and let W_x denote the stabilizer of x under the action of W . To x we assign the root subsystem

$$\Gamma_x := \Gamma_{W_x} \equiv \{\alpha \in \Sigma : \sigma_{\alpha}(x) = x\}. \quad (10)$$

Due to (8),

$$\Gamma_{w(x)} = w(\Gamma_x) \quad \forall w \in W. \quad (11)$$

Hence, to the W -orbit of x we can associate the conjugacy class of Γ_x under the action of W on Σ . This conjugacy class will be referred to as the reflection type of x . It will be denoted by \mathbf{r}_x .

Now, let \mathbf{R} denote the set of conjugacy classes of root subsystems and let $\mathbf{R}^* \subseteq \mathbf{R}$ denote the subset of reflection types. For $\mathbf{r} \in \mathbf{R}$, let $T_{\mathbf{r}}^{\mathbb{C}}$ denote the subset of $T^{\mathbb{C}}$ of points having reflection type \mathbf{r} and let $\mathcal{P}_{\mathbf{r}}$ to be the subset of \mathcal{P} of orbits having reflection type \mathbf{r} . Obviously, $\mathcal{P}_{\mathbf{r}}$ is the image of $T_{\mathbf{r}}^{\mathbb{C}}$ under the natural projection $\text{pr} : T^{\mathbb{C}} \rightarrow \mathcal{P}$. The subsets $T_{\mathbf{r}}^{\mathbb{C}}$ and $\mathcal{P}_{\mathbf{r}}$ are nonempty precisely iff $\mathbf{r} \in \mathbf{R}^*$. In this case, we will refer to them as the reflection type subsets of $T^{\mathbb{C}}$ and \mathcal{P} , respectively. As a result, we obtain disjoint decompositions

$$T^{\mathbb{C}} = \bigcup_{\mathbf{r} \in \mathbf{R}^*} T_{\mathbf{r}}^{\mathbb{C}}, \quad \mathcal{P} = \bigcup_{\mathbf{r} \in \mathbf{R}^*} \mathcal{P}_{\mathbf{r}}.$$

The set \mathbf{R} is partially ordered by inclusion modulo conjugacy, i.e., $\mathbf{r}_1 \leq \mathbf{r}_2$ if there exist representatives Γ_i of \mathbf{r}_i such that $\Gamma_1 \subseteq \Gamma_2$. Let

$$T_{(\mathbf{r})}^{\mathbb{C}} := \{x \in T^{\mathbb{C}} : \mathbf{r}_x \geq \mathbf{r}\}, \quad \mathcal{P}_{(\mathbf{r})} := \text{pr}(T_{(\mathbf{r})}^{\mathbb{C}}).$$

$T_{(\mathbf{r})}^{\mathbb{C}}$ consists of the points whose stabilizer contains a subgroup which is generated by a representative of \mathbf{r} . We have the disjoint decompositions

$$T_{(\mathbf{r})}^{\mathbb{C}} = \bigcup_{\mathbf{r}' \geq \mathbf{r}} T_{\mathbf{r}'}^{\mathbb{C}}, \quad \mathcal{P}_{(\mathbf{r})} = \bigcup_{\mathbf{r}' \geq \mathbf{r}} \mathcal{P}_{\mathbf{r}'}.$$

5 Relations for reflection types

In this section, we derive relations characterizing the subset $T_{(\mathbf{r})}^{\mathbb{C}}$ inside $T^{\mathbb{C}}$ for all $\mathbf{r} \in \mathbf{R}$. In a first step, we determine the root subsystems Γ_x associated with the elements of $T^{\mathbb{C}}$. Let $x \in T^{\mathbb{C}}$. Let $a \in G$ and $B \in \mathfrak{t}$ such that $x = a \exp(iB)$ and choose $A \in \mathfrak{t}$ such that $a = \exp(A)$. Then, for $\alpha \in \Sigma$, we have $\sigma_{\alpha}(x) = x$ iff

$$\exp(\sigma_{\alpha}(A)) \exp(i\sigma_{\alpha}(B)) = \exp(A) \exp(iB).$$

By uniqueness of the polar decomposition, this is equivalent to

$$\exp(\sigma_\alpha(A)) = \exp(A) \quad \text{and} \quad \sigma_\alpha(B) = B.$$

By (6), the second equation holds iff

$$\alpha(B) = 0.$$

The first equation holds iff

$$\sigma_\alpha(A) - A \in \ker(\exp).$$

To analyse this condition, choose a system of generators $\{H_i\}$ of the lattice $\ker(\exp)$ and expand

$$\sigma_\alpha(A) - A = \sum_i k_i H_i \tag{12}$$

with integers k_i . Since every root is contained in a base, we can find a base Π containing α . Since $\{\frac{4\pi i}{k(\beta, \beta)} H_\beta : \beta \in \Pi\}$ is a basis in \mathfrak{t} and since the \mathbb{Z} -lattice spanned by this basis is contained in the lattice $\ker(\exp) \subset \mathfrak{t}$, there exist rationals $r_{i\beta}$, $i = 1, \dots, \text{rank}(G)$, $\beta \in \Pi$, such that

$$H_i = \sum_\beta r_{i\beta} \frac{4\pi i}{k(\beta, \beta)} H_\beta, \quad i = 1, \dots, \text{rank}(G).$$

Plugging this and (6) into (12), we obtain the system of equations

$$2\pi i \sum_i r_{i\alpha} k_i = -\alpha(A), \tag{13}$$

$$2\pi i \sum_i r_{i\beta} k_i = 0 \quad \forall \beta \in \Pi \setminus \{\alpha\}. \tag{14}$$

Lemma 5.1. *There exist relatively prime integers p_α, q_α such that the subset*

$$\left\{ \sum_i r_{i\alpha} k_i : (k_1, \dots, k_r) \text{ is an integer solution of (14)} \right\} \tag{15}$$

of \mathbb{R} coincides with $\frac{p_\alpha}{q_\alpha} \mathbb{Z}$.

Proof. Denote the subset (15) by M_α . Since the integer solutions of (14) form a group, M_α is a subgroup of \mathbb{R} . Since there exists an integer m_α such that $m_\alpha r_{\beta i}$ is an integer for all i , M_α is discrete and hence a lattice in \mathbb{R} . Now, the existence theorem for lattice bases, see e.g. [33, Thm. IX.1.1], yields $y_\alpha \in \mathbb{R}$ such that $M_\alpha = y_\alpha \mathbb{Z}$. Since M_α is contained in the rationals, we finally obtain $y_\alpha = \frac{p_\alpha}{q_\alpha}$ for relatively prime integers p_α, q_α . \square

Thus, we have found that $\sigma_\alpha(x) = x$ iff $\alpha(A) \in 2\pi i \frac{p_\alpha}{q_\alpha} \mathbb{Z}$ and $\alpha(B) = 0$. Since $\alpha(B)$ is purely imaginary, this condition can be rewritten in the form

$$e^{\frac{q_\alpha}{p_\alpha} \alpha(A + iB)} = 1. \tag{16}$$

Recall that every functional $\eta \in (\mathfrak{t}^\mathbb{C})^*$ with the property $\eta(\ker(\exp)) \subset 2\pi i\mathbb{Z}$ descends to a smooth function $e_\eta : T^\mathbb{C} \rightarrow \mathbb{C}$ via

$$e_\eta(\exp(X)) = e^{\eta(X)}. \quad (17)$$

The functional $\frac{q_\alpha}{p_\alpha}\alpha$ is of this type: since $\ker(\exp) \subset \mathfrak{t}$ and since it is invariant under σ_α , every $X \in \ker(\exp)$ satisfies $\sigma_\alpha(X) - X \in \ker(\exp)$ and hence $\frac{q_\alpha}{p_\alpha}\alpha(X) \in 2\pi i\mathbb{Z}$. Therefore, the function $e_{\frac{q_\alpha}{p_\alpha}\alpha}$ exists. Define

$$E_\alpha := e_{\frac{q_\alpha}{p_\alpha}\alpha} - 1.$$

Using (16) and the fact that $x = \exp(A + iB)$, we finally conclude that $\sigma_\alpha(x) = x$ iff $E_\alpha(x) = 0$. This proves

Proposition 5.2. *For $x \in T^\mathbb{C}$, one has $\Gamma_x = \{\alpha \in \Sigma : E_\alpha(x) = 0\}$.* \square

Remark 5.3. If G is simply connected, $\ker(\exp)$ coincides with the lattice generated by the elements $2\pi i \frac{2H_\beta}{k(\beta, \beta)}$, where β belongs to some base in Σ . As a consequence, the lattice (15) coincides with \mathbb{Z} and $p_\alpha = q_\alpha = 1$. Hence, in this case, $E_\alpha = e_\alpha - 1$. \blacklozenge

In a second step, we derive relations characterizing the subsets $T_{(\mathbf{r})}^\mathbb{C}$ in terms of the functions E_α . For $\mathbf{r} \in \mathbf{R}$, choose a representative Γ and define a function $\nu_{\mathbf{r}}$ on $T^\mathbb{C}$ by

$$\nu_{\mathbf{r}} := \sum_{w \in W} \prod_{\alpha \in \Sigma \setminus w(\Gamma)} E_\alpha. \quad (18)$$

This function does obviously not depend on the choice of Γ .

Lemma 5.4. *For all $x \in T^\mathbb{C}$, we have $\nu_{\mathbf{r}_x}(x) \neq 0$.*

Proof. We can choose $\Gamma = \Gamma_x$ in the definition of $\nu_{\mathbf{r}_x}$. Then, by Proposition 5.2, $E_\alpha(x) = 0$ for all $\alpha \in \Gamma$ and $E_\alpha(x) \neq 0$ for all $\alpha \notin \Gamma$. Hence, in the sum over $w \in W$, every term with $w(\Gamma) = \Gamma$ gives a contribution

$$\prod_{\alpha \in \Sigma \setminus w(\Gamma)} E_\alpha(x) = \prod_{\alpha \in \Sigma \setminus \Gamma} E_\alpha(x)$$

which is nonzero and independent of w , whereas the terms with $w(\Gamma) \neq \Gamma$ do not contribute, because $\Sigma \setminus w(\Gamma)$ contains some $\alpha \in \Gamma$ here. Consequently, $\nu_{\mathbf{r}_x}(x)$ is a multiple of a nonzero complex number. \square

Theorem 5.5. *Let $\mathbf{r}_0 \in \mathbf{R}$ and $x \in T^\mathbb{C}$. Then, $x \in T_{(\mathbf{r}_0)}^\mathbb{C}$ iff $\nu_{\mathbf{r}}(x) = 0$ for all $\mathbf{r} \not\geq \mathbf{r}_0$.*

As a consequence,

$$T_{(\mathbf{r}_0)}^\mathbb{C} = \bigcap_{\mathbf{r} \not\geq \mathbf{r}_0} \nu_{\mathbf{r}}^{-1}(0). \quad (19)$$

Proof. First, assume that $x \in T_{(\mathbf{r}_0)}^{\mathbb{C}}$. Let $\mathbf{r} \in \mathbf{R}$ such that $\mathbf{r} \not\geq \mathbf{r}_0$ and choose a representative Γ of \mathbf{r} to compute $\nu_{\mathbf{r}}$ according to (18). Since $\mathbf{r}_x \geq \mathbf{r}_0$, we have $\mathbf{r} \not\geq \mathbf{r}_x$. This means that for any $w \in W$, Γ_x is not a subset of $w(\Gamma)$ and hence $\Sigma \setminus w(\Gamma)$ contains an element of Γ_x . Thus, Proposition 5.2 implies that $\nu_{\mathbf{r}}(x) = 0$. Conversely, assume that $\nu_{\mathbf{r}}(x) = 0$ for all $\mathbf{r} \in \mathbf{R}$ such that $\mathbf{r} \not\geq \mathbf{r}_0$. Then, Lemma 5.4 implies $\mathbf{r}_x \geq \mathbf{r}_0$ and hence $x \in T_{(\mathbf{r}_0)}^{\mathbb{C}}$. \square

Theorem 5.5 yields the following characterization of $T_{\mathbf{r}}^{\mathbb{C}}$.

Corollary 5.6. *Let $\mathbf{r}_0 \in \mathbf{R}$ and $x \in T^{\mathbb{C}}$. Then, $x \in T_{\mathbf{r}_0}^{\mathbb{C}}$ iff $\nu_{\mathbf{r}_0}(x) \neq 0$ and $\nu_{\mathbf{r}}(x) = 0$ for all $\mathbf{r} \not\geq \mathbf{r}_0$.*

Proof. If $x \in T_{\mathbf{r}_0}^{\mathbb{C}}$, then Theorem 5.5 implies that $\nu_{\mathbf{r}}(x) = 0$ for all $\mathbf{r} \not\geq \mathbf{r}_0$ and Lemma 5.4 implies that $\nu_{\mathbf{r}_0}(x) = \nu_{\mathbf{r}_x}(x) \neq 0$. Conversely, if $\nu_{\mathbf{r}}(x) = 0$ for all $\mathbf{r} \not\geq \mathbf{r}_0$, the theorem yields $x \in T_{(\mathbf{r}_0)}^{\mathbb{C}}$. If, in addition, $\nu_{\mathbf{r}_0}(x) \neq 0$, then x cannot belong to $T_{(\mathbf{r}')}^{\mathbb{C}}$ for any $\mathbf{r}' > \mathbf{r}_0$, because $\mathbf{r}' > \mathbf{r}_0$ implies $\mathbf{r}_0 \not\geq \mathbf{r}'$, so that every $y \in T_{(\mathbf{r}')}^{\mathbb{C}}$ satisfies $\nu_{\mathbf{r}_0}(y) = 0$. Thus, $x \in T_{(\mathbf{r}_0)}^{\mathbb{C}} \setminus \left(\bigcup_{\mathbf{r}' > \mathbf{r}_0} T_{(\mathbf{r}')}^{\mathbb{C}} \right) = T_{\mathbf{r}_0}^{\mathbb{C}}$. \square

Remark 5.7. Let $\mathbf{r}, \mathbf{r}' \in \mathbf{R}$ and assume that \mathbf{r}' is a direct successor of \mathbf{r} . According to Theorem 5.5, the subset $T_{(\mathbf{r}')}^{\mathbb{C}}$ of $T_{(\mathbf{r})}^{\mathbb{C}}$ is the set of solutions of the relations $\nu_{\mathbf{r}} = 0$ and $\nu_{\mathbf{r}''} = 0$ for all $\mathbf{r}'' > \mathbf{r}$ such that $\mathbf{r}'' \not\geq \mathbf{r}'$ (that is, all successors of \mathbf{r} which cannot be compared with \mathbf{r}'). Of course, there might be further reflection types to be taken into account to define $T_{(\mathbf{r}')}^{\mathbb{C}}$ as a subset of $T^{\mathbb{C}}$. From the slice theorem we know that strata whose orbit types are not comparable are separated, i.e., they admit open neighbourhoods which do not intersect. This carries over to reflection type subsets, because the decomposition by orbit types is at least as fine as that by reflection types. Thus, the relation $\nu_{\mathbf{r}} = 0$ defines $T_{(\mathbf{r})}^{\mathbb{C}}$ locally, i.e., as a subset of some open subset of $T_{(\mathbf{r})}^{\mathbb{C}}$, whereas the relations $\nu_{\mathbf{r}''} = 0$ for all $\mathbf{r}'' > \mathbf{r}$ such that $\mathbf{r}'' \not\geq \mathbf{r}'$ are necessary to define $T_{(\mathbf{r}')}^{\mathbb{C}}$ globally as a subset of $T_{(\mathbf{r})}^{\mathbb{C}}$. In particular, they do not reduce the dimension. Note that the relation $\nu_{\mathbf{r}} = 0$ reduces the dimension iff \mathbf{r} is infinitesimal, cf. Remark 8.2 below. \blacklozenge

In a third step, we express the functions $\nu_{\mathbf{r}}$ in terms of the normalized characters $\tilde{\chi}_{\lambda}^{\mathbb{C}}$ of $G^{\mathbb{C}}$, see Section 2. For that purpose, we choose a Weyl chamber in $(\mathfrak{t}^{\mathbb{C}})^*$ and denote the corresponding sets of positive roots and of dominant weights by Σ_+ and Λ_+ , respectively. Recall that for every functional $\eta \in (\mathfrak{t}^{\mathbb{C}})^*$ with the property $\eta(\ker(\exp)) \subset 2\pi i\mathbb{Z}$, the function e_{η} is defined by (17). Let

$$A_{\eta} := \sum_{w \in W} \text{sign}(w) e_{w\eta}. \quad (20)$$

Let $\mathbf{r} \in \mathbf{R}$. We expand

$$\nu_{\mathbf{r}} = \sum_{\lambda \in \Lambda_+} C_{\lambda}^{\mathbf{r}} \tilde{\chi}_{\lambda}^{\mathbb{C}}.$$

Since restriction to G yields

$$(\nu_{\mathbf{r}})|_G = \sum_{\lambda \in \Lambda_+} \frac{C_{\lambda}^{\mathbf{r}}}{N_{\lambda}} \chi_{\lambda},$$

we can compute the coefficients $C_\lambda^{\mathbf{r}}$ via the integrals

$$C_\lambda^{\mathbf{r}} = N_\lambda \int_G \overline{\chi_\lambda} \nu_{\mathbf{r}} d\mu.$$

By Weyl's integration formula,

$$C_\lambda^{\mathbf{r}} = \frac{N_\lambda}{|W|} \int_T \overline{\chi_\lambda} \nu_{\mathbf{r}} |A_\delta|^2 d\mu,$$

where δ denotes half the sum of the positive roots, given by (4), $|W|$ denotes the cardinality (order) of W and $d\mu_T$ denotes the normalized Haar measure on T . Using Weyl's character formula,

$$\chi_\lambda = \frac{A_{\lambda+\delta}}{A_\delta},$$

we can rewrite this as

$$C_\lambda^{\mathbf{r}} = \frac{N_\lambda}{|W|} \int_T \overline{A_{\lambda+\delta}} A_\delta \nu_{\mathbf{r}} d\mu_T. \quad (21)$$

To evaluate the integral, we choose a root subsystem Γ representing \mathbf{r} and expand

$$\nu_{\mathbf{r}} = \sum_{w \in W} \prod_{\alpha \in \Sigma \setminus w(\Gamma)} E_{\frac{q_\alpha}{p_\alpha} \alpha} = \sum_{w \in W} \prod_{\alpha \in \Sigma \setminus \Gamma} \left(e_{\frac{q_\alpha}{p_\alpha} w\alpha} - 1 \right) = \sum_{w \in W} \sum_{\Delta \subseteq \Sigma \setminus \Gamma} (-1)^{|\Sigma \setminus \Gamma| - |\Delta|} \prod_{\alpha \in \Delta} e_{\frac{q_\alpha}{p_\alpha} w\alpha},$$

where $|\cdot|$ denotes the cardinality. Since both Σ and Γ are root systems, the cardinality of $\Sigma \setminus \Gamma$ is even. Thus, using $e_{\eta_1} e_{\eta_2} = e_{\eta_1 + \eta_2}$, as well as the notation

$$\lambda_\Delta := \sum_{\alpha \in \Delta} \frac{q_\alpha}{p_\alpha} \alpha,$$

with $\lambda_\emptyset = 0$ understood, we obtain

$$\nu_{\mathbf{r}} = \sum_{w \in W} \sum_{\Delta \subseteq \Sigma \setminus \Gamma} (-1)^{|\Delta|} e_{w\lambda_\Delta}. \quad (22)$$

Plugging this and (20) into (21) and using that on G one has $\overline{e_\eta} = e_{-\eta}$, we arrive at

$$C_\lambda^{\mathbf{r}} = \frac{N_\lambda}{|W|} \sum_{w, w', w''} \sum_{\Delta \subseteq \Sigma \setminus \Gamma} (-1)^{|\Delta|} \text{sign}(w') \text{sign}(w'') \int_T e_{w\lambda_\Delta + w'\delta - w''(\lambda + \delta)} d\mu_T$$

The integral vanishes unless $w\lambda_\Delta + w'\delta - w''(\lambda + \delta) = 0$, in which case it evaluates to 1. Hence, we can write

$$C_\lambda^{\mathbf{r}} = \frac{N_\lambda}{|W|} \sum_{w, w', w''} \sum_{\Delta \subseteq \Sigma \setminus \Gamma} (-1)^{|\Delta|} \text{sign}(w') \text{sign}(w'') \delta_{w\lambda_\Delta, w''(\lambda + \delta) - w'\delta},$$

where upright δ denotes the Kronecker symbol. By shifting the summation over w' by w^{-1} and the summation over w'' by $w^{-1}w'^{-1}$, and by carrying out the sum over w , we obtain

$$C_\lambda^{\mathbf{r}} = N_\lambda \sum_{w', w''} \sum_{\Delta \subseteq \Sigma \setminus \Gamma} (-1)^{|\Delta|} \text{sign}(w'') \delta_{\lambda_\Delta, w'(w''(\lambda + \delta) - \delta)}.$$

In effect, the sum over Δ counts the number of subsets $\Delta \subseteq \Sigma \setminus \Gamma$ which satisfy $\lambda_\Delta = w'(w''(\lambda + \delta) - \delta)$, weighted by a factor $(-1)^{|\Delta|}$. Thus, for $\eta \in (\mathfrak{t}^\mathbb{C})^*$, let $V_\Gamma^+(\eta)$ denote the number of subsets $\Delta \subseteq \Sigma \setminus \Gamma$ which satisfy $\lambda_\Delta = \eta$ and which have an even number of elements (including the empty set) and let $V_\Gamma^-(\eta)$ denote the corresponding number of subsets which have an odd number of elements. Define

$$V_\Gamma(\eta) = V_\Gamma^+(\eta) - V_\Gamma^-(\eta).$$

Then,

$$C_\lambda^\mathbf{r} = N_\lambda \sum_{w', w'' \in W} \text{sign}(w'') V_\Gamma(w'(w''(\lambda + \delta) - \delta)). \quad (23)$$

Let $\Lambda_+^\mathbf{r} \subset \Lambda_+$ denote the subset consisting of elements λ such that $C_\lambda^\mathbf{r} \neq 0$.

Proposition 5.8. *For $\mathbf{r} \in \mathbf{R}$,*

$$\nu_\mathbf{r} = \sum_{\lambda \in \Lambda_+^\mathbf{r}} C_\lambda^\mathbf{r} \tilde{\chi}_\lambda^\mathbb{C}, \quad C_\lambda^\mathbf{r} = N_\lambda \sum_{w', w'' \in W} \text{sign}(w'') V_\Gamma(w'(w''(\lambda + \delta) - \delta)),$$

where Γ is a root subsystem representing \mathbf{r} . □

The coefficients $C_\lambda^\mathbf{r}$ represent $\nu_\mathbf{r}$ under the identification of \mathcal{H} with the Hilbert space ℓ^2 over the index set Λ_+ , see Section 2. They can be easily computed using computer algebra, see Section 8. This concludes the discussion of the relations characterizing the subsets $T_{(\mathbf{r})}^\mathbb{C} \subseteq T^\mathbb{C}$.

Remark 5.9. Depending on Γ , the computation of the coefficients $C_\lambda^\mathbf{r}$ can be simplified by rewriting the sum over w' in (23) as follows. Let

$$W_\Gamma := \{w \in W : w(\Gamma) = \Gamma\}.$$

If $w'(w''(\lambda + \delta) - \delta) = \lambda_\Delta$ and $w \in W_\Gamma$, then $ww'(w''(\lambda + \delta) - \delta) = \lambda_{w\Delta}$, where $w\Delta \subset \Sigma \setminus \Gamma$, and $V_\Gamma(ww'(w''(\lambda + \delta) - \delta)) = V_\Gamma(w'(w''(\lambda + \delta) - \delta))$. Hence, using the notation

$$\tilde{V}_\Gamma(\eta) := \sum_{w' \in W/W_\Gamma} V_\Gamma(w'(\eta)),$$

where the sum runs over a selection of one representative for each right W_Γ -coset, Formula (23) can be rewritten as $C_\lambda^\mathbf{r} = |W_\Gamma| \tilde{C}_\lambda^\mathbf{r}$ with the reduced coefficients

$$\tilde{C}_\lambda^\mathbf{r} = N_\lambda \sum_{w'' \in W} \text{sign}(w'') \tilde{V}_\Gamma(w''(\lambda + \delta) - \delta) \quad (24)$$

and instead of the functions $\nu_\mathbf{r}$ one may work with the reduced functions $\nu_\mathbf{r}/|W_\Gamma|$. The multiplicities $\tilde{V}_\Gamma(\eta)$ count the number of ways of writing η in the form $\eta = w'^{-1}(\lambda_\Delta)$ with $w' \in W/W_\Gamma$ and with Δ being a subset of $\Sigma \setminus \Gamma$ with an even number of elements minus this number with Δ having an odd number of elements. ◆

Example 5.10. Let us illustrate the formula for $C_\lambda^\mathbf{r}$ given in Proposition 5.8 for $G = \mathrm{SU}(2)$. The root system is $\Sigma = \{\alpha, -\alpha\}$, the Weyl group is $W = \{1, -1\}$, acting by multiplication, and $\delta = \frac{\alpha}{2}$. Dominant weights are of the form $\lambda_k = \frac{k}{2}\alpha$ with a nonnegative integer k and

$$w'(w''(\lambda_k + \delta) - \delta) = \begin{cases} \pm \frac{k}{2}\alpha & | \ w' = \pm 1, \ w'' = 1, \\ \pm \frac{k+2}{2}\alpha & | \ w' = \pm 1, \ w'' = -1. \end{cases}$$

Since $\mathrm{SU}(2)$ is simply connected, $q_\alpha = p_\alpha = 1$. There are two root subsystems, $\Gamma = \emptyset$ and $\Gamma = \Sigma$. We have $T_{([\emptyset])}^\mathbb{C} = T^\mathbb{C}$ and $T_{([\Sigma])}^\mathbb{C} = T_{[\Sigma]}^\mathbb{C}$. Since $[\emptyset] \not\geq [\Sigma]$, the subset $T_{[\Sigma]}^\mathbb{C} \subset T^\mathbb{C}$ is defined by the relation $\nu_{[\emptyset]} = 0$. To determine $\nu_{[\emptyset]}$, we calculate λ_Δ for all subsets $\Delta \subseteq \Sigma \setminus \emptyset = \Sigma$:

$$\lambda_\Delta = \begin{cases} \alpha & \Delta = \{\alpha\}, \\ -\alpha & \Delta = \{-\alpha\}, \\ 0 & \Delta = \emptyset \text{ or } \{\alpha, -\alpha\}. \end{cases}$$

Hence,

$$V_\emptyset(\pm\alpha) = -1, \quad V_\emptyset(0) = 2$$

and $V_\emptyset(\eta) = 0$ for all other η . For given k , we compute

$$w'(w''(\lambda_k + \delta) - \delta) = w' \frac{w''(k+1) - 1}{2} \alpha = \begin{cases} \pm \frac{k}{2}\alpha & | \ w' = \pm 1, \ w'' = 1, \\ \pm \frac{k+2}{2}\alpha & | \ w' = \pm 1, \ w'' = -1. \end{cases}$$

Hence, taking the sum over w', w'' in the order $(w', w'') = (1, 1), (-1, 1), (1, -1), (-1, -1)$, we obtain

$$\frac{C_{\lambda_0}^{[\emptyset]}}{N_{\lambda_0}} = V_\Gamma(0) + V_\Gamma(0) - V_\Gamma(-\alpha) - V_\Gamma(\alpha) = 6, \quad \frac{C_{\lambda_2}^{[\emptyset]}}{N_{\lambda_2}} = V_\Gamma(\alpha) + V_\Gamma(-\alpha) = -2.$$

All other coefficients vanish. Since $\Gamma = \emptyset$ is invariant under all of W , according to Remark 5.9, the corresponding reduced coefficients are

$$\frac{\tilde{C}_{\lambda_0}^0}{N_{\lambda_0}} = 3, \quad \frac{\tilde{C}_{\lambda_2}^0}{N_{\lambda_2}} = -1.$$

Let us add that under the adjoint action of $\mathrm{SU}(2)$, the stabilizers of the elements of $T_{[\emptyset]}^\mathbb{C}$ coincide with the center and the stabilizers of the elements of $T_{[\Sigma]}^\mathbb{C}$ coincide with $\mathrm{SU}(2)$. Hence, in the present case, reflection types and orbit types are equivalent. This is true for all $\mathrm{SU}(n)$, see Remark 8.2. \blacklozenge

6 Costratification

In this section, we sketch how to use the relations derived above to construct the subspaces $\mathcal{H}_\mathbf{r}$ associated with the reflection types.

Consider a chosen reflection type \mathbf{r}_0 . According to Theorem 5.5, then $T_{(\mathbf{r}_0)}^{\mathbb{C}}$ is defined by the relations $\nu_{\mathbf{r}} = 0$ for all $\mathbf{r} \not\geq \mathbf{r}_0$. For every $\mathbf{r} \in \mathbf{R}$, multiplication by $\nu_{\mathbf{r}}$ defines a linear operator

$$K^{\mathbf{r}} : \mathcal{H} \rightarrow \mathcal{H}, \quad K^{\mathbf{r}}\psi = \nu_{\mathbf{r}}\psi,$$

which can be easily seen to be bounded and to have closed range. Then,

$$\mathcal{V}_{\mathbf{r}_0} = \sum_{\mathbf{r} \not\geq \mathbf{r}_0} K^{\mathbf{r}}(\mathcal{H})$$

and, consequently,

$$\mathcal{H}_{\mathbf{r}_0} = \bigcap_{\mathbf{r} \not\geq \mathbf{r}_0} \ker((K^{\mathbf{r}})^{\dagger}). \quad (25)$$

Consider the matrix elements

$$K_{\lambda'\lambda}^{\mathbf{r}} = \langle \tilde{\chi}_{\lambda'}^{\mathbb{C}}, K^{\mathbf{r}} \tilde{\chi}_{\lambda}^{\mathbb{C}} \rangle$$

of $K^{\mathbf{r}}$ with respect to the orthonormal basis of normed complex characters $\tilde{\chi}_{\lambda}^{\mathbb{C}}$. Since $\nu_{\mathbf{r}}$, as well as any product of normed complex characters, is a *real* linear combination of normed complex characters, the matrix elements $K_{\lambda'\lambda}^{\mathbf{r}}$ are real. In terms of these matrix elements, the condition that a function $\psi = \sum_{\lambda \in \Lambda_+} H_{\lambda} \tilde{\chi}_{\lambda}^{\mathbb{C}}$ belongs to $\mathcal{H}_{\mathbf{r}_0}$ reads

$$\sum_{\lambda' \in \Lambda_+} K_{\lambda'\lambda}^{\mathbf{r}} H_{\lambda'} = 0 \quad \forall \lambda \in \Lambda_+, \mathbf{r} \not\geq \mathbf{r}_0. \quad (26)$$

To compute the matrix elements $K_{\lambda'\lambda}^{\mathbf{r}}$, we use Proposition 5.8 and the expansion

$$\tilde{\chi}_{\lambda'}^{\mathbb{C}} \tilde{\chi}_{\lambda}^{\mathbb{C}} = \sum_{\lambda'' \in \Lambda_+} M_{\lambda'\lambda\lambda''} \tilde{\chi}_{\lambda''}^{\mathbb{C}} \quad (27)$$

to obtain

$$K_{\lambda'\lambda}^{\mathbf{r}} = \sum_{\lambda'' \in \Lambda_+^{\mathbf{r}}} C_{\lambda''}^{\mathbf{r}} M_{\lambda'\lambda\lambda''}. \quad (28)$$

The coefficients $C_{\lambda''}^{\mathbf{r}}$ are given by Proposition 5.8. For the coefficients $M_{\lambda'\lambda\lambda''}$, the representation theory of semisimple Lie algebras yields the following. By restricting the expansion (27) to G and renaming the labels $\lambda', \lambda, \lambda''$, we obtain the equation

$$\chi_{\lambda'} \chi_{\lambda} = \sum_{\lambda'' \in \Lambda_+} \frac{N_{\lambda''} N_{\lambda}}{N_{\lambda'}} M_{\lambda''\lambda\lambda'} \chi_{\lambda'}.$$

For $\lambda'' \in \Lambda_+$, let $G(\lambda'')$ denote the weight system of the corresponding irrep. For a weight $\mu \in G(\lambda'')$ of this irrep, let $m_{\lambda''}(\mu)$ denote the multiplicity. Finally, for $\lambda, \lambda' \in \Lambda_+$ and a weight μ of an arbitrary irrep, there exists at most one $w \in W$ such that

$$w(\lambda + \mu + \delta) = \lambda' + \delta. \quad (29)$$

Let $\tau_{\lambda\lambda'}(\mu) = \text{sign}(w)$ in case such w exists and $\tau_{\lambda\lambda'}(\mu) = 0$ otherwise. Then,

$$M_{\lambda''\lambda\lambda'} = \frac{N_{\lambda'}}{N_{\lambda}N_{\lambda''}} \sum_{\mu \in G(\lambda'')} m_{\lambda''}(\mu) \tau_{\lambda\lambda'}(\mu).$$

Since the multiplicities $m_{\lambda''}(\mu)$ are invariant under W and since every W -orbit in $G(\lambda'')$ intersects Λ_+ in exactly one point [26], the sum over $\mu \in G(\lambda'')$ may be split as follows. Let

$$G_+(\lambda'') := G(\lambda'') \cap \Lambda_+$$

be the subset of dominant weights in the irrep λ'' . For a dominant weight $\mu \in G_+(\lambda'')$, define

$$T_{\lambda\lambda'}(\mu) := \sum_{\mu' \in W(\mu)} \tau_{\lambda\lambda'}(\mu'),$$

where $W(\mu)$ denotes the W -orbit of μ . Then,

$$M_{\lambda''\lambda\lambda'} = \frac{N_{\lambda'}}{N_{\lambda}N_{\lambda''}} \sum_{\mu \in G_+(\lambda'')} m_{\lambda''}(\mu) T_{\lambda\lambda'}(\mu).$$

Plugging this into (28), we obtain

$$K_{\lambda'\lambda}^{\mathbf{r}} = \frac{N_{\lambda'}}{N_{\lambda}} \sum_{\lambda'' \in \Lambda_+^{\mathbf{r}}} \frac{C_{\lambda''}^{\mathbf{r}}}{N_{\lambda''}} \sum_{\mu \in G_+(\lambda'')} m_{\lambda''}(\mu) T_{\lambda\lambda'}(\mu).$$

To separate the irregular factor $T_{\lambda\lambda'}(\mu)$, we interchange the order of summation. Let

$$G_+^{\mathbf{r}} := \bigcup_{\lambda'' \in \Lambda_+^{\mathbf{r}}} G_+(\lambda'')$$

(the dominant weights of all the irreps in $\Lambda_+^{\mathbf{r}}$). Then,

$$K_{\lambda'\lambda}^{\mathbf{r}} = \frac{N_{\lambda'}}{N_{\lambda}} \sum_{\mu \in G_+^{\mathbf{r}}} D_{\mu}^{\mathbf{r}} T_{\lambda\lambda'}(\mu) \quad \text{with} \quad D_{\mu}^{\mathbf{r}} := \sum_{\lambda'' \in \Lambda_+^{\mathbf{r}}} \frac{C_{\lambda''}^{\mathbf{r}}}{N_{\lambda''}} m_{\lambda''}(\mu), \quad (30)$$

with $m_{\lambda''}(\mu) = 0$ if $\mu \notin G(\lambda'')$ understood. The factors $D_{\mu}^{\mathbf{r}}$ can be easily determined from Proposition 5.8 and Freudenthal's formula for the multiplicities $m_{\lambda''}(\mu)$, see the examples in Section 8. The factors $T_{\lambda\lambda'}(\mu)$ are irregular and have to be determined case by case. However, if λ is sufficiently far from the boundary of the chosen dominant Weyl chamber, these factors become very simple. Let us call $\lambda \in \Lambda_+$ stable if $\lambda + \mu$ belongs to the dominant Weyl chamber for all $\mu \in G_+^{\mathbf{r}}$. For stable λ and arbitrary λ' , there exists $w \in W$ such that (29) holds iff $\lambda + \mu' = \lambda'$. Thus, $\tau_{\lambda\lambda'}(\mu') = \delta_{\lambda+\mu',\lambda'}$ and hence

$$T_{\lambda\lambda'}(\mu) = \begin{cases} 1 & | \lambda' - \lambda \in W(\mu), \\ 0 & | \text{otherwise.} \end{cases}$$

Plugging this into (30) and using that the W -orbits of distinct elements of $G_+^{\mathbf{r}}$ do not intersect, we obtain the simple formula

$$K_{\lambda+w(\mu),\lambda}^{\mathbf{r}} = \begin{cases} \frac{N_{\lambda+w(\mu)}}{N_{\lambda}} D_{\mu}^{\mathbf{r}} & | \mu \in G_+^{\mathbf{r}}, w \in W, \\ 0 & | \text{otherwise,} \end{cases} \quad (31)$$

which holds for stable $\lambda \in \Lambda_+$.

Remark 6.1. All of the above can be worked out equally well by means of the reduced functions $\nu_{\mathbf{r}}/|W_{\Gamma}|$, discussed in Remark 5.9. This leads to a reduced operator $\tilde{K}^{\mathbf{r}} = K^{\mathbf{r}}/|W_{\Gamma}|$ and to reduced coefficients $\tilde{D}_{\mu}^{\mathbf{r}} = D_{\mu}^{\mathbf{r}}/|W_{\Gamma}|$. Formulae (30) and (31) remain true if one replaces $K^{\mathbf{r}}$ by $\tilde{K}^{\mathbf{r}}$, $C_{\lambda}^{\mathbf{r}}$ by $\tilde{C}_{\lambda}^{\mathbf{r}}$ and $K^{\mathbf{r}}$ by $\tilde{K}_{\lambda}^{\mathbf{r}}$. \blacklozenge

Example 6.2. To illustrate Formula (30) for the coefficients $D_{\mu}^{\mathbf{r}}$, consider $G = \mathrm{SU}(2)$. Recall from Example 5.10 that the only root subsystem different from Σ is $\Gamma = \emptyset$, defining the generic reflection type $\mathbf{r} = 0$. We use the notation and the results of that example. The elements of Λ_+^0 are $\lambda'' = \lambda_0, \lambda_2$. The dominant weights of the corresponding irreps are λ_0 for $\lambda'' = \lambda_0$ and $0, \lambda_2$ for $\lambda'' = \lambda_2$. The multiplicities of these weights are 1. Hence, (30) yields

$$D_{\lambda_0}^0 = \frac{C_{\lambda_0}^0}{N_{\lambda_0}} m_{\lambda_0}(\lambda_0) + \frac{C_{\lambda_2}^0}{N_{\lambda_2}} m_{\lambda_2}(\lambda_0) = 4, \quad D_{\lambda_2}^0 = \frac{C_{\lambda_2}^0}{N_{\lambda_2}} m_{\lambda_2}(\lambda_2) = -2.$$

Since Γ is invariant under all of W and $|W| = 2$, the corresponding reduced coefficients are $\tilde{D}_{\lambda_0}^0 = 2$, $\tilde{D}_{\lambda_2}^0 = -1$. \blacklozenge

7 Root subsystems

To complete the discussion of the reflection type decomposition, we still have to find the set \mathbf{R} of W -conjugacy classes of root subsystems of Σ . The classification of root subsystems up to the action of W or, equivalently, the classification of reflection subgroups of W up to conjugacy under inner automorphisms, can be found in [8, 9, 12]. See §1.3 of [9] for historical remarks.

Following [9], we say that a root subsystem Γ of Σ is closed if for all $\alpha, \beta \in \Gamma$ such that $\alpha + \beta \in \Sigma$ one has $\alpha + \beta \in \Gamma$. Beware that in [12], the term root subsystem is used to refer to what is here called a closed root subsystem. Obviously, the linear subspace

$$\mathfrak{g}_{\Gamma}^{\mathbb{C}} := \mathrm{span}_{\mathbb{C}}\{H_{\alpha} : \alpha \in \Gamma\} \oplus \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_{\alpha}^{\mathbb{C}},$$

of $\mathfrak{g}^{\mathbb{C}}$ is a Lie subalgebra iff Γ is closed. Because of this connection with Lie subalgebras and Lie subgroups, the closed root subsystems have been classified up to conjugacy under W long ago by Borel and de Siebenthal [2] and Dynkin [10]. Their analysis is based on the following observation. Recall that a subset $\Pi \subset \Sigma$ is called a base (or a simple system) of Σ if it is a basis in $(\mathfrak{t}^{\mathbb{C}})^*$ with respect to which the elements of Σ have either positive

or negative integer coefficients. Given a closed root subsystem Γ and a base Π in Γ , let Π' be obtained by extending Π by the negative of the highest (with respect to Π) root of Γ . Then, every proper subset of Π' is a base of a closed root subsystem of Γ , and hence the base of a closed root subsystem of Σ . This way, up to conjugacy under W , all the closed root subsystems of Σ can be generated iteratively from a base in Σ . The remaining, non-closed, root subsystems arise by a similar algorithm [5, 8, 9, 12]: as usual, define the dual roots by

$$\alpha^\vee := \frac{2\alpha}{k(\alpha, \alpha)}, \quad \alpha \in \Sigma.$$

Given Γ and Π , let Π'' be obtained by extending Π by the negative of the dual of the highest (with respect to Π^\vee) root of the dual root system Γ^\vee . Then, every proper subset of Π'' is a base of a root subsystem of Γ , and hence the base of a root subsystem of Σ . Up to conjugacy under W , the root subsystems generated iteratively in this way from a base of Σ include all the non-closed root subsystems of Σ . Since Σ^\vee differs from Σ only if it contains roots of different lengths, this implies in particular that non-closed root subsystems can exist only in root systems containing a factor B_l , C_l , F_4 or G_2 . In contrast, all the root subsystems of A_l , D_l , E_6 , E_7 and E_8 are closed.

For the classical series, the algorithm described above yields the following result. Given a base $\Pi = \{\alpha_1, \dots, \alpha_n\}$ in Σ , denote

$$\begin{aligned} \alpha_l^A &:= \alpha_l + \dots + \alpha_n, & \alpha_l^B &:= \alpha_l + 2(\alpha_{l+1} + \dots + \alpha_n), \\ \alpha_l^C &:= 2(\alpha_l + \dots + \alpha_{n-1}) + \alpha_n, & \tilde{\alpha}_l^C &:= \alpha_l + 2(\alpha_{l+1} + \dots + \alpha_{n-1}) + \alpha_n, \end{aligned}$$

where $1 \leq l < n$ and

$$\alpha_l^D := \alpha_l + 2(\alpha_{l+1} + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n,$$

where $1 \leq l < n - 2$. Given a sequence (i_1, \dots, i_n) , let (i^1, \dots, i^n) denote the sequence of partial sums, i.e., $i^1 = i_1$, $i^2 = i_1 + i_2$ etc.

Proposition 7.1. *The following list contains the W -conjugacy classes of root subsystems of the classical series and a base of a representative for each class. Assume $n \geq 1$ for A_n , $n \geq 2$ for B_n , C_n and $n \geq 4$ for D_n , as well as $i_l, k_l \geq 1$ and $j_l \geq 2$ for all l .*

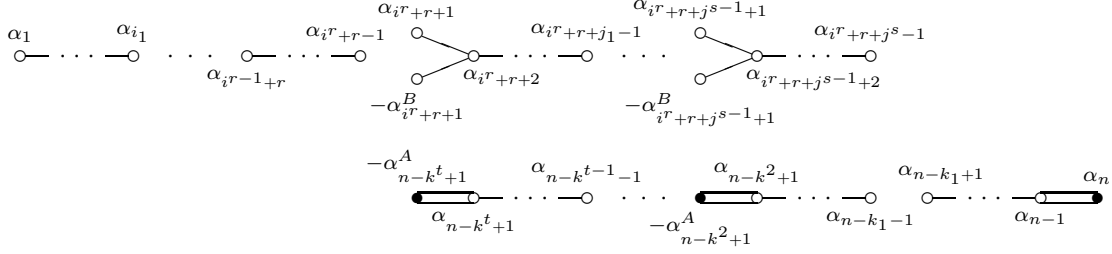
(A_n) *Classes:* $A_{i_1} \oplus \dots \oplus A_{i_r}$, where $r \geq 0$, $i_1 \leq \dots \leq i_r$ and $i^r + r - 1 \leq n$.

$$\text{Bases: } \overset{\alpha_1}{\circ} \dots \overset{\alpha_{i_1}}{\circ} \overset{\alpha_{i_1+2}}{\circ} \dots \overset{\alpha_{i_2+1}}{\circ} \dots \overset{\alpha_{i^{r-1}+r}}{\circ} \dots \overset{\alpha_{i^r+r-1}}{\circ}$$

All root subsystems are closed.

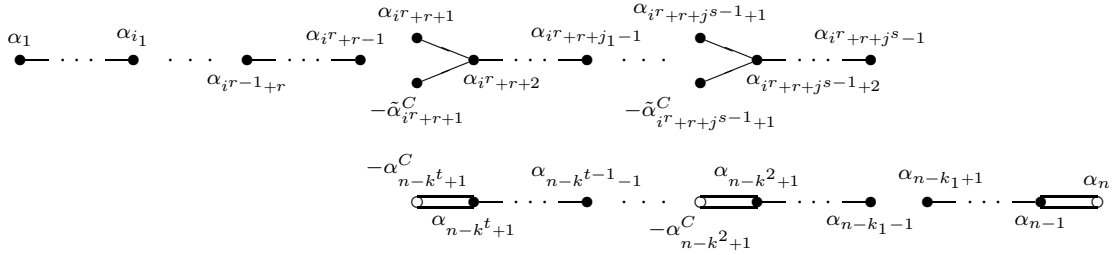
(B_n) *Classes:* $A_{i_1} \oplus \dots \oplus A_{i_r} \oplus D_{j_1} \oplus \dots \oplus D_{j_s} \oplus B_{k_1} \oplus \dots \oplus B_{k_t}$, where $r, s, t \geq 0$, $i_1 \leq \dots \leq i_r$, $j_1 \leq \dots \leq j_s$, $k_1 \leq \dots \leq k_{t-1}$ and $i^r + r + j^s + k^t \leq n$.

Bases:



Closed root subsystems are those with $t \leq 1$.

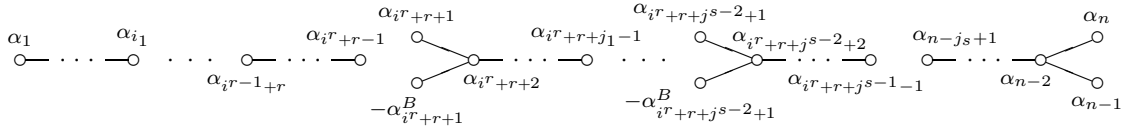
(C_n) Classes: $A_{i_1} \oplus \dots \oplus A_{i_r} \oplus D_{j_1} \oplus \dots \oplus D_{j_s} \oplus C_{k_1} \oplus \dots \oplus C_{k_t}$, where $r, s, t \geq 0$, $i_1 \leq \dots \leq i_r$, $j_1 \leq \dots \leq j_s$, $k_1 \leq \dots \leq k_{t-1}$ and $i^r + r + j^s + k^t \leq n$.



Closed root subsystems are those with $s = 0$, i.e., without D -factors.

(D_n) Classes: $A_{i_1} \oplus \dots \oplus A_{i_r} \oplus D_{j_1} \oplus \dots \oplus D_{j_s}$, where $r, s \geq 0$, $i_1 \leq \dots \leq i_r$, $j_1 \leq \dots \leq j_s$ and $i^r + r + j^s \leq n$.

Bases:



All root subsystems are closed.

In case $k_l = 1$, the factor B_{k_l} or C_{k_l} is isomorphic to A_1 and the corresponding piece of the Dynkin diagram shrinks to \bullet or \circ , respectively. Similarly, in case $j_l = 2$, the factor D_{j_l} is isomorphic to $A_1 \oplus A_1$ and the corresponding piece of the Dynkin diagram shrinks to \circ or \bullet . Nevertheless, while being isomorphic, root subsystems obtained from one another by replacing a factor B_1 or C_1 by A_1 , a factor D_2 by $A_1 \oplus A_1$ or a factor D_3 by A_3 are not conjugate under W .

Proof. We assume that the elements of the base Π of Σ are numbered in such a way that the subset $\{\alpha_1, \dots, \alpha_{n-1}\}$ is of class A_{n-1} . Starting from Π and passing from lower to higher indices, we successively split off the necessary factors, always keeping the part $\tilde{\Pi}$ of Π which contains α_n . While A -factors can just be split off by removing one element from $\tilde{\Pi}$, before splitting off the other factors, one has to extend $\tilde{\Pi}$ by either the negative of its highest root or the negative of the dual of the highest root of $\tilde{\Pi}^\vee$. An elementary computation yields the following. Let $\tilde{\Pi} = \{\alpha_l, \dots, \alpha_n\}$.

1. For B_n , the highest root of $\tilde{\Pi}$ is α_l^B and the dual of the highest root of $\tilde{\Pi}^\vee$ is α_l^A .

2. For C_n , the highest root of $\tilde{\Pi}$ is α_l^C and the dual of the highest root of $\tilde{\Pi}^\vee$ is $\tilde{\alpha}_l^C$.
3. For D_n , the highest root of $\tilde{\Pi}$ is α_l^D .

Now, the assertion follows from the results in [9] and [10]. \square

Figure 1 shows the Hasse diagrams of the set \mathbf{R} of W -conjugacy classes of root subsystems for some examples. For each of the bases given in Proposition 7.1, one can easily generate the corresponding root subsystem as a subset of Σ by means of standard routines of the computer algebra system GAP [32] with the package CHEVIE [14]. This is necessary for the computation of the coefficients $C_\lambda^{\mathbf{r}}$ and $D_\lambda^{\mathbf{r}}$, see Section 8.

After having determined the set \mathbf{R} of conjugacy classes of root subsystems, in order to find the reflection type decomposition, from the subsets $T_{\mathbf{r}}^{\mathbb{C}}$, $\mathbf{r} \in \mathbf{R}$, one still has to single out those which are nonempty. In the case where G is simply connected, this effort can be reduced by the following observation.

Proposition 7.2. *If G is simply connected, then Γ_x is closed for all $x \in T^{\mathbb{C}}$.*

Thus, if G is simply connected, one can restrict attention to closed root subsystems. Of course, this does not make a difference unless $\mathfrak{g}^{\mathbb{C}}$ contains a factor B_l , C_l , F_4 or G_2 .

Proof. Let $\alpha, \beta \in \Gamma_x$ such that $\alpha + \beta \in \Sigma$. Since G is simply connected, we have $q_\alpha = p_\alpha = 1$ and $q_\beta = p_\beta = 1$. Therefore, Proposition 5.2 yields $e_\alpha(x) = e_\beta(x) = 1$ and hence

$$E_{\alpha+\beta}(x) = e_{\alpha+\beta}(x) - 1 = e_\alpha(x)e_\beta(x) - 1 = 0,$$

i.e., $\alpha + \beta \in \Gamma_x$. \square

If G is not simply connected, the argument fails, because here one has $E_\alpha = e_{\frac{q_\alpha}{p_\alpha}}\alpha - 1$ and there is no simple relation between the ratios $\frac{q_\alpha}{p_\alpha}$ for the elements of Γ and the corresponding ratios of their sums. This is illustrated by the following example.

Example 7.3. Consider $G = \mathrm{SO}(5)$ and its (twofold) universal covering group $\tilde{G} = \mathrm{Sp}(2) \cong \mathrm{Spin}(5)$. Every base of the root system Σ consists of a short root α and a long root β and

$$\Sigma = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta)\},$$

see Figure 2(a). The Killing form can be chosen so that $k(\alpha, \alpha) = 1$. Then,

$$k(\alpha, \beta) = -1, \quad k(\beta, \beta) = 2, \quad k(\alpha + \beta, \alpha + \beta) = 1, \quad k(2\alpha + \beta, 2\alpha + \beta) = 2.$$

The elements $2\pi i \frac{2H_\gamma}{k(\gamma, \gamma)}$, $\gamma \in \Sigma$, of \mathfrak{t} are thus given, respectively, by

$$\pm 4\pi i H_\alpha, \quad \pm 2\pi i H_\beta, \quad \pm 4\pi i H_{\alpha+\beta}, \quad \pm 2\pi i H_{2\alpha+\beta}$$

and $\ker(\exp_{\tilde{G}})$ is generated by $4\pi i H_\alpha$ and $2\pi i H_\beta$, whereas $\ker(\exp_G)$ is generated by $2\pi i H_\alpha$ and $2\pi i H_\beta$, see Figure 2(b). Consider the subsets of long roots and of short roots,

$$\Gamma_l = \{\pm\beta, \pm(2\alpha + \beta)\}, \quad \Gamma_s = \{\pm\alpha, \pm(\alpha + \beta)\}.$$

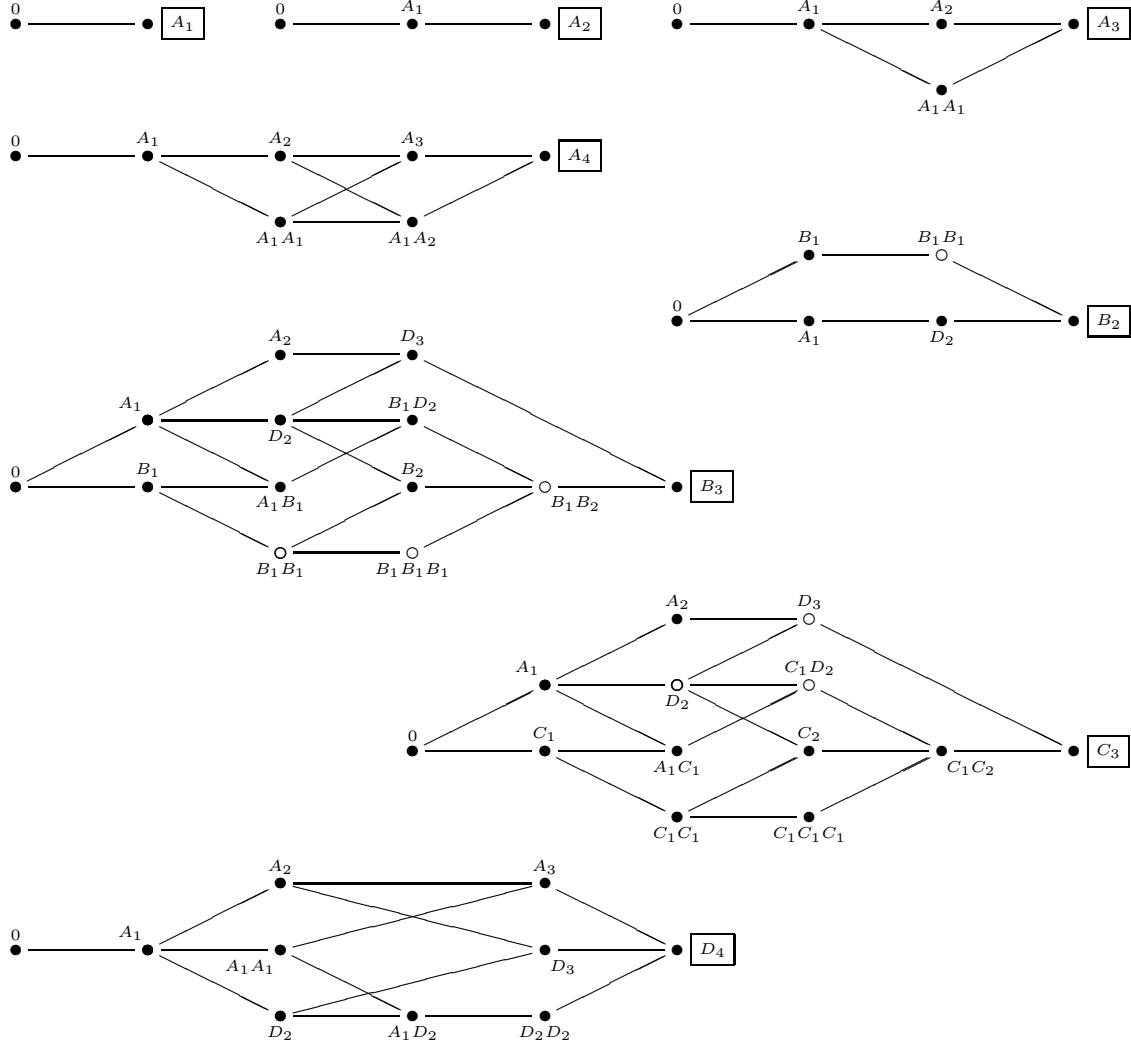


Figure 1: Hasse diagrams of the sets of W -conjugacy classes of root subsystems for some low rank classical semisimple Lie algebras, cf. Proposition 7.1. Closed root subsystems are represented by dots, non-closed ones by circles.

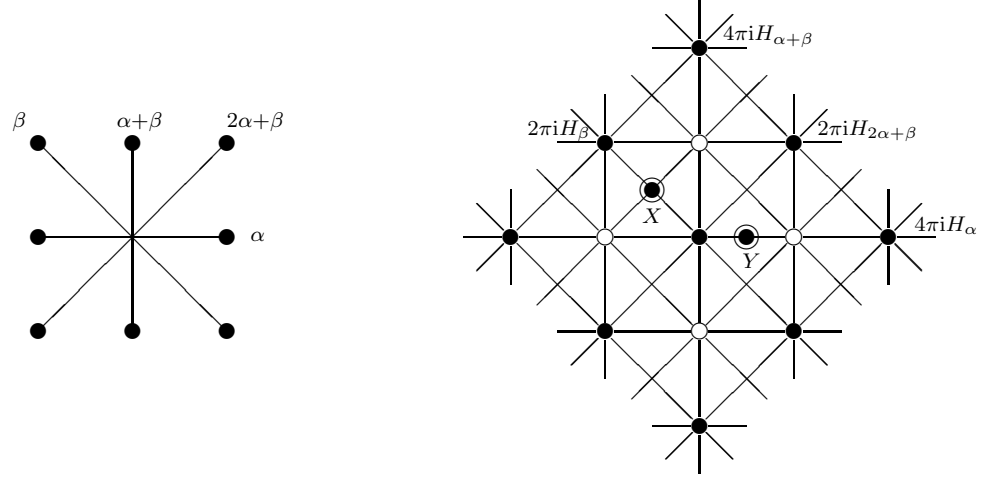


Figure 2: Root system Σ (a) and Cartan subalgebra \mathfrak{t} (b) of Example 7.3. In (b), the elements of $\ker(\exp_{\tilde{G}})$ and of $\ker(\exp_G) \setminus \ker(\exp_{\tilde{G}})$ are represented by dots and circles, respectively.

Both of them are root subsystems. They represent the classes $C_1 \oplus C_1$ and D_2 , respectively, of Proposition 7.1. While Γ_l is closed, Γ_s is not. Hence, only Γ_l may represent a reflection type for \tilde{G} . It does, indeed: for $X = \pi i H_\beta$, from Figure 2(b) we read off

$$\begin{aligned} \sigma_\alpha(X) - X &= 2\pi i H_\alpha, & \sigma_\beta(X) - X &= -2\pi i H_\beta, \\ \sigma_{\alpha+\beta}(X) - X &= -2\pi i H_{\alpha+\beta}, & \sigma_{2\alpha+\beta}(X) - X &= 0. \end{aligned}$$

Since $-2\pi i H_\beta$ belongs to $\ker(\exp_{\tilde{G}})$, whereas $2\pi i H_\alpha$ and $-2\pi i H_{\alpha+\beta}$ do not, we obtain $\Gamma_{\exp_{\tilde{G}} X} = \Gamma_l$. For G , things are different: for $Y = \pi i H_\alpha$, from Figure 2(b) we read off

$$\begin{aligned} \sigma_\alpha(Y) - Y &= -2\pi i H_\alpha, & \sigma_\beta(Y) - Y &= \pi i H_\beta, \\ \sigma_{\alpha+\beta}(Y) - Y &= 0, & \sigma_{2\alpha+\beta}(Y) - Y &= -\pi i H_{2\alpha+\beta}. \end{aligned}$$

Since $-2\pi i H_\alpha$ belongs to $\ker(\exp_G)$, whereas $\pi i H_\beta$ and $-\pi i H_{2\alpha+\beta}$ do not, $\Gamma_{\exp_G Y} = \Gamma_s$. Thus, Γ_s represents a reflection type for G . \blacklozenge

Remark 7.4.

By analogy, one can define reflection types for the action of W on $\mathfrak{t}^\mathbb{C}$ induced from the adjoint representation of G on \mathfrak{g} . Let us refer to these reflection types as infinitesimal reflection types. Since \exp is locally a diffeomorphism, it is clear that every infinitesimal reflection type is a reflection type of the action of W on $T^\mathbb{C}$. Conversely, since $\eta \in \mathfrak{t}^\mathbb{C}$ is invariant under a reflection σ_α iff $\alpha(\eta) = 0$, the reflection type defined by a closed root subsystem Γ of Σ is infinitesimal iff there is no larger root subsystem $\Gamma' \supset \Gamma$ of Σ such that $\text{span}_\mathbb{C} \Gamma = \text{span}_\mathbb{C} \Gamma'$. This means that there is no larger root subsystem of the same

rank. Thus, for A_n , all reflection types are infinitesimal, whereas all the other classical series admit reflection types which are not infinitesimal, e.g., $C_{n-1} \oplus C_1$ for C_n with $n \geq 2$, D_n for B_n with $n \geq 3$ and $D_2 \oplus D_{n-2}$ for D_n with $n \geq 4$. In particular, the reflection types discussed in Example 7.3 are not infinitesimal, because they have maximal rank. ♦

8 Examples

As a test, we computed the coefficients $\tilde{C}_\lambda^{\mathbf{r}}$ and $\tilde{D}_\mu^{\mathbf{r}}$ for all conjugacy classes of closed root subsystems for the classical Lie algebras up to rank 4 using the computer algebra system GAP [32] with the package CHEVIE [14] and Mathematica [35] with the package LieART [11]. According to Proposition 7.2, the closed root subsystems yield all candidates for the reflection types of the corresponding simply connected compact Lie groups. We proceeded as follows. Let \mathbf{r} be one of the conjugacy classes of closed root subsystems from Proposition 7.1. To start with, we use GAP and CHEVIE to generate a list **COMPLEMENT** of the elements of $\Sigma \setminus \Gamma$ from the corresponding base given in that Proposition. Using **COMPLEMENT**, we compile a list **WEYL2** containing one representative of each W_Γ -coset in W . Then, we generate a list **SUBSETSUM1** of all pairs $(\eta, V_\Gamma(\eta))$, where $\eta = \lambda_\Delta$ for some subset of $\Delta \subset \Sigma \setminus \Gamma$, as follows. Starting with the single pair $(\eta, V) = (0, 1)$, in step n , for each pair (η, V) of the list from step $n - 1$, we append to this list a new pair $(\eta + \mathbf{COMPLEMENT}[n], -V)$. From the enlarged list so produced we obtain the list of step n by collecting pairs with the same η and taking the sum over all the associated V 's. This procedure circumvents the generation of all subsets of $\Sigma \setminus \Gamma$. Next, we generate a list **SUBSETSUM2** of all pairs $(\eta, \tilde{V}_\Gamma(\eta))$, where $\eta = w'^{-1}\lambda_\Delta$ for some subset $\Delta \subset \Sigma \setminus \Gamma$ and the chosen representative w' of some right W_Γ -coset in W . Starting with **SUBSETSUM1**, in step n , we take the n -th element w' of **WEYL2** and successively check for each (η, V) in the list of step $n - 1$ whether $w'^{-1}(\eta) = \eta'$ for some pair (η', V') . If so, we replace V' by $V' + V$ in this pair. If not, we add the pair $(w'^{-1}(\eta), V)$ to the list. Finally, we produce a list **IRREP** of all $\lambda \in \Lambda_+$ satisfying

$$\|\lambda + \delta\| \leq M + \|\delta\|, \quad (32)$$

where M is the maximum of the norms $\|\eta\|$ over all pairs (η, V) in **SUBSETSUM1**. This list contains all elements of $\Lambda_+^{\mathbf{r}}$, because for every subset $\Delta \subset \Sigma \setminus \Gamma$, the relation $w'(w''(\lambda + \delta) - \delta) = \lambda_\Delta$ implies (32). Now, using **SUBSETSUM2**, for every member λ of **IRREP**, we compute the quotient $\tilde{C}_\lambda^{\mathbf{r}}/N_\lambda$ of the reduced coefficient $\tilde{C}_\lambda^{\mathbf{r}}$ by means of Formula (24). By omitting members whose coefficient vanishes, we finally obtain a list **COEFF** containing all pairs $(\lambda, \tilde{C}_\lambda^{\mathbf{r}}/N_\lambda)$ with $\lambda \in \Lambda_+^{\mathbf{r}}$.

The computation of the reduced coefficients $\tilde{D}_\mu^{\mathbf{r}}$ is straightforward. First, using the routine **DominantWeightSystem** of LieART, we generate a list **MULT** of triples $(\lambda, \mu, m_\lambda(\mu))$, where $\lambda \in \Lambda_+^{\mathbf{r}}$ and $\mu \in G_+(\lambda)$. From this, we extract a list of the elements μ of $G_+^{\mathbf{r}}$, i.e., the dominant weights of all the irreps contained in $\Lambda_+^{\mathbf{r}}$. Using **COEFF**, for each member of this list, we compute the reduced coefficient $\tilde{D}_\mu^{\mathbf{r}}$.

Tables with the results are given in the appendix. For lack of space, we omit the results for B_4 and C_4 .

Remark 8.1. We compare our results with those of [20], where the case of $G = \mathrm{SU}(n+1)$ was discussed. Let $\lambda_1, \dots, \lambda_n$ denote the fundamental weights of A_n . Using the mapping

$$\varphi := (-\chi_{\lambda_1}^{\mathbb{C}}, \chi_{\lambda_2}^{\mathbb{C}}, \dots, (-1)^n \chi_{\lambda_n}^{\mathbb{C}}, (-1)^{n+1}) : T^{\mathbb{C}} \rightarrow \mathbb{C}^{n+1},$$

embedding $T^{\mathbb{C}}$ into the space of normalized complex polynomials of degree $n+1$ with constant coefficient being equal to $(-1)^{n+1}$, it was shown there that for $k = 1, \dots, n$, the subset $T_{(A_k)}^{\mathbb{C}}$ of $T_{(A_{k-1})}^{\mathbb{C}}$ (in our notation and with $T_{(A_0)}^{\mathbb{C}} = T^{\mathbb{C}}$ understood) is defined by a relation of the type $\bar{D} \circ \varphi = 0$, where D is the discriminant of a certain polynomial of degree $n+2-k$. For example, denoting the discriminant of the polynomial $a_0 z^n + a_1 z^{n-1} + \dots + a_n$ by $D_n(a_0, \dots, a_n)$, the relations are

$$D_2(1, -\chi_{\lambda_1}, 1) = 0 \quad \text{defining } T_{(A_1)}^{\mathbb{C}} \subset T^{\mathbb{C}}, \quad (33)$$

for $n = 1$,

$$D_3(1, -\chi_{\lambda_1}, \chi_{\lambda_2}, -1) = 0 \quad \text{defining } T_{(A_1)}^{\mathbb{C}} \subset T^{\mathbb{C}}, \quad (34)$$

$$D_2(3, -2\chi_{\lambda_1}, \chi_{\lambda_2}) = 0 \quad \text{defining } T_{(A_2)}^{\mathbb{C}} \equiv T_{A_2}^{\mathbb{C}} \subset T_{(A_1)}^{\mathbb{C}} \quad (35)$$

for $n = 2$ and

$$D_4(1, -\chi_{\lambda_1}, \chi_{\lambda_2}, -\chi_{\lambda_3}, 1) = 0 \quad \text{defining } T_{(A_1)}^{\mathbb{C}} \subset T^{\mathbb{C}}, \quad (36)$$

$$D_3(4, -3\chi_{\lambda_1}, 2\chi_{\lambda_2}, -\chi_{\lambda_3}) = 0 \quad \text{defining } T_{(A_2)}^{\mathbb{C}} \subset T_{(A_1)}^{\mathbb{C}}, \quad (37)$$

$$D_2(12, -6\chi_{\lambda_1}, 2\chi_{\lambda_2}) = 0 \quad \text{defining } T_{(A_3)}^{\mathbb{C}} \equiv T_{A_3}^{\mathbb{C}} \subset T_{(A_2)}^{\mathbb{C}}. \quad (38)$$

for $n = 3$. A direct computation shows that (33), (34) and (36) coincide with the relation $\nu_0 = 0$ in the respective case and that $\nu_{A_1} = 0$ for $n = 2$ coincides with

$$D_2(3, -2\chi_{\lambda_1}, \chi_{\lambda_2}) D_2(3, -2\chi_{\lambda_2}, \chi_{\lambda_1}) = 0,$$

which is equivalent to (35). The other relations are not directly comparable, because they have different meanings. For example, while the relation (37) means that solutions in $T_{(A_1)}^{\mathbb{C}}$ *do* belong to the subset $T_{(A_2)}^{\mathbb{C}}$, the relation $\nu_{A_1} = 0$ means that solutions in $T_{(A_1)}^{\mathbb{C}}$ *do not* belong to $T_{A_1}^{\mathbb{C}}$ and thus belong to the rest, which is $T_{(A_2)}^{\mathbb{C}} \cup T_{(A_1 \oplus A_1)}^{\mathbb{C}}$. \blacklozenge

Remark 8.2.

It remains to analyze the precise difference between the decompositions by orbit types and by reflection types. We do not discuss this here. Let us just state that a direct inspection yields that in the cases $G = \mathrm{SU}(n+1)$ and $G = \mathrm{Sp}(n)$, reflection types correspond bijectively to orbit types and, therefore, the reflection type subsets coincide with the orbit type subsets. Indeed, the stabilizers under the adjoint action of G are given by the

Howe subgroups³ of G containing a maximal torus. By [31], these subgroups are given as follows. For $G = \mathrm{SU}(n+1)$, they are labelled by a set of positive integers $\{k_1, \dots, k_r\}$ such that $\sum_{i=1}^r k_i = n+1$. The class labelled by such a set is represented by the subgroup

$$H = \left\{ \mathrm{diag}(a_1, \dots, a_r) : a_i \in \mathrm{U}(k_i), \prod_{i=1}^r \det(a_i) = 1 \right\},$$

acting on $\mathbb{C}^{k_1} \oplus \dots \oplus \mathbb{C}^{k_r}$. For $G = \mathrm{Sp}(n)$, they are labelled by two sets of positive integers $\{k_1, \dots, k_r\}, \{l_1, \dots, l_s\}$ such that $\sum_{i=1}^r k_i + \sum_{i=1}^s l_i = n$ and the class labelled by such a pair of sets is represented by the subgroup

$$H = \{ \mathrm{diag}(a_1, \dots, a_r, b_1, \dots, b_s) : a_i \in \mathrm{U}(k_i), b_i \in \mathrm{Sp}(l_i) \},$$

acting on $\mathbb{H}^{k_1} \oplus \dots \oplus \mathbb{H}^{k_r} \oplus \mathbb{H}^{l_1} \oplus \dots \oplus \mathbb{H}^{l_s}$. To H there corresponds the subgroup

$$W_H := (H \cap \mathrm{N}_G(T))/T$$

of W , acting on $\mathfrak{t}^{\mathbb{C}}$. In the case $G = \mathrm{SU}(n+1)$, we choose T to consist of the diagonal matrices in G . Then, $\mathfrak{t}^{\mathbb{C}}$ consists of the diagonal matrices in $\mathfrak{sl}(n+1, \mathbb{C})$ and W_H consists of the elements $\pi_1 \times \dots \times \pi_r$, where π_i is a permutation of k_i elements and π_1 acts on the first k_1 entries, π_2 acts on the entries number $k_1 + 1$ up to $k_1 + k_2$ etc. If we further choose the elements $\alpha_i, i = 1, \dots, n$, of the base Π_0 to be given by the difference between the $(i+1)$ th and the i th entry, then W_H is generated by the reflections associated with the simple roots $\alpha_{k(i-1)+1}, \dots, \alpha_{k(i)}$ for all i such that $k_i \geq 2$. Thus, W_H represents the reflection type labelled by the regular semisimple subalgebra $A_{k_1-1} \oplus \dots \oplus A_{k_r-1}$, where factors $A_0 \equiv 0$ can be omitted.

In the case $G = \mathrm{Sp}(n)$, we write quaternionic $(n \times n)$ -matrices as complex $(2n \times 2n)$ -matrices by replacing the quaternion q by the (2×2) -matrix

$$\begin{bmatrix} \overline{q_1} & -\overline{q_2} \\ q_2 & q_1 \end{bmatrix},$$

where $q = q_1 + q_2 j$ with $q_1, q_2 \in \mathbb{C}$ and j denoting the second quaternionic unit. Then, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(n, \mathbb{C})$. In analogy with the case $G = \mathrm{SU}(n+1)$, by choosing a concrete representation of T and a concrete base Π_0 , one can check that W_H represents the reflection type labelled by $A_{k_1-1} \oplus \dots \oplus A_{k_r-1} \oplus C_{l_1} \oplus \dots \oplus C_{l_s}$, where factors $A_0 \equiv 0$ can be omitted again. \blacklozenge

Acknowledgments

The authors acknowledge discussions with Erik Fuchs, Florian Fürstenberg, Michael Schellenberger, Christian Fleischhack and Rainer Matthes.

³subgroups $H \subseteq G$ satisfying $\mathrm{C}_G^2(H) = H$

A Tables

In the following tables, dominant weights $\lambda \in \Lambda_+$ are given by their Dynkin labels $\frac{2k(\lambda, \alpha_i)}{k(\alpha_i, \alpha_i)}$. For each closed root subsystem, the first column contains the reduced coefficient $\tilde{C}_\lambda^r/N_\lambda$ and the second column contains the reduced coefficient \tilde{D}_λ^r .

SU(2) (A_1)

λ	0
0	3 2
2	-1 -1

SU(3) (A_2)

λ	0	A_1
00	15 6	20 12
03	3 2	1 1
11	-6 -2	-5 -3
22	-1 -1	
30	3 2	1 1

SU(4) (A_3)

λ	0	A_1	$A_1 \oplus A_1$	A_2
000	105 24	350 120	105 54	50 32
004	-15 -6	-20 -12		-1 -1
012	27 4	54 14		3 2
020	-6 4	-14 16	21 16	-8 -4
032	-3 -2	-1 -1		
040	9 4	6 4	1 1	
101	-45 -6	-120 -27	-35 -12	-6 -4
113	6 2	5 3		
121	-12 -2	-16 -5	-3 -2	
202	-9 -4	-15 -12	5 3	
210	27 4	54 14		3 2
222	1 1			
230	-3 -2	-1 -1		
303	-3 -2	-1 -1		
311	6 2	5 3		
400	-15 -6	-20 -12		-1 -1

Sp(2) (C_2)

λ	0	C_1	A_1	$C_1 \oplus C_1$
00	21 8	21 12	28 16	5 4
01	-3 -2	-7 -4	6 -1	-3 -2
02	-3	-4 -2	3 4	
03	-3 -2		-1 -1	
20	-6 -2		-14 -8	1 1
21	6 2	3 2	2 1	
22	1 1			
40	-3 -2	-1 -1		

SU(5) (A_4)

λ	0	A_1	$A_1 \oplus A_1$	A_2	$A_1 \oplus A_2$	A_3
0000	945 120	6300 1200	6720 1920	1899 740	1520 880	105 80
0005	105 24	350 120	105 54	50 32		1 1
0013	-180 -12	-750 -84	-230 -39	-132 -40	10 4	-3 -2
0021	45 -12	210 -96	-250 -189	198 32	2 -12	3 1
0042	-15 -6	-20 -12		-1 -1		
0050	45 12	75 30	20 12	3 2	1 1	
0102	270 12	1350 102	945 120	165 40	-45 -2	6 4
0110	-75 12	-400 108	225 195	-285 16	168 80	-21 -7
0123	27 4	54 14		3 2		
0131	-54 -4	-135 -18	-45 -12	-6 -2	-3 -2	
0204	-6 4	-14 16	21 16	-8 -4		
0212	-27 -8	-81 -44	-18 -20	15 4	3 1	
0220	111 8	370 48	90 0	6 4	12 8	
0301	-18 4	-66 26	25 36	-9 6	-7 5	
0322	-3 -2	-1 -1				
0330	9 4	6 4	1 1			
0403	9 4	6 4	1 1			
0411	-18 -4	-21 -8	-5 -3			
0500	45 12	75 30	20 12	3 2	1 1	
1001	-420 -24	-2450 -228	-2520 -360	-382 -106	-411 -141	-4
1014	-45 -6	-120 -27	-35 -12	-6 -4		
1022	81 4	270 24	135 27	9 0	6 4	
1030	-18 4	-66 26	25 36	-9 6	-7 5	
1103	72 8	264 52	-10 12	33 20	-13 -5	
1111	-144 -2	-600 -15	-220 6	-108 -26	-22 -15	
1132	6 2	5 3				
1140	-18 -4	-21 -8	-5 -3			
1200	45 -12	210 -96	-250 -189	198 32	2 -12	3 1
1213	-12 -2	-16 -5	-3 -2			
1221	24 2	44 7	18 6			
1302	18 4	39 16	2 7			
1310	-54 -4	-135 -18	-45 -12	-6 -2	-3 -2	
2002	-90 -18	-420 -144	-175 -144	-6 -28	128 72	
2010	270 12	1350 102	945 120	165 40	-45 -2	6 4
2023	-9 -4	-15 -12	5 3			
2031	18 4	39 16	2 7			
2104	27 4	54 14		3 2		
2112	-36 0	-96 0	-37 -12	-9 -4		
2120	-27 -8	-81 -44	-18 -20	15 4	3 1	
2201	81 4	270 24	135 27	9 0	6 4	
2222	1 1					
2230	-3 -2	-1 -1				
2303	-3 -2	-1 -1				
2311	6 2	5 3				
2400	-15 -6	-20 -12		-1 -1		
3003	-36 -12	-114 -66	56 24	-6 -10	1 1	
3011	72 8	264 52	-10 12	33 20	-13 -5	
3032	-3 -2	-1 -1				
3040	9 4	6 4	1 1			
3100	-180 -12	-750 -84	-230 -39	-132 -40	10 4	-3 -2
3113	6 2	5 3				
3121	-12 -2	-16 -5	-3 -2			
3202	-9 -4	-15 -12	5 3			
3210	27 4	54 14		3 2		
4004	-15 -6	-20 -12		-1 -1		
4012	27 4	54 14		3 2		
4020	-6 4	-14 16	21 16	-8 -4		
4101	-45 -6	-120 -27	-35 -12	-6 -4		
5000	105 24	350 120	105 54	50 32		1 1

$\text{Sp}(3) (C_3)$

λ	0		A_1		C_1		$A_1 \oplus C_1$		A_2		$C_1 \oplus C_1$		C_2		$C_1 \oplus C_2$	
000	231	48	1232	384	462	144	924	432	511	360	275	120	81	60	54	54
002	-18		18	12	-66	-6	126	48	-56	-64	-77	-24	-10	-3		
004	15	6	20	12					1	1						
010	-42	-8	-14	-52	-168	-28	-210	-74	224	-12	-154	-28	-17	-8	-27	-12
012	-24		6	16	-59	-8	-24	-6		8	-12	-2				
020	-36	-4	-84		-84	-24	-154	-56	168	112	-15	-22	-9	-2	5	3
022	-24	-4	-42	-12	-11	-4	-6	-4	-3	-2						
030	-33	-4	-158	-26	-9	-6	66	45	-7	-12	-10	-8	1			
032	3	2	1	1												
040	-3	-4	26		-18	-12		-4	4	2	-4	-2				
050	-9	-4	-3	-2	-3	-2	-1	-1								
101	24	6	-34	35	105	20	168	56	-112	8	88	20	-10	-4		
103	9		3	-4	12	2	10	3		-2						
111	24	-2	20	-25	60	2	-66	-25		-8	36	8	8	2		
113	-6	-2	-5	-3												
121	27	2	54	7	27	4		-1	3	2	3					
131	12	2	10	3	6	2	3	2								
200	-63	-8	-462	-96	-21	-4	-154	-48	-357	-200	33	12	15	9	15	16
202	15	2	-7	-8	36	12	18	12	8	8	11	4				
210	57	4	324	38	28	4	84	18	21	12	-30	-6	9	6	-3	-2
212	3	2	13	9	-3		-3	-1								
220		6	36	40	-18	7	-6		-24	-12		6	-3	-1		
222	-1	-1														
230	-9		-39	-12	6	6	-4	-3			3	2				
240	3	2			1	1										
301	-54	-4	-144	-10	-72	-14	-18	-5	7	-2	-28	-12	-6	-4		
303	3	2	1	1												
311	-18	-4	-15	-10	-24	-8	4	1			-6	-2				
321	-6	-2	-2	-1	-3	-2										
400	-27	-8	-162	-80					36	20	21	16	-3	-6	1	1
402	3		-3	-4	4	2										
410	36	4	72	14	42	8					9	2	3	2		
420	6	2	14	8							-1	-1				
501	3	2	-6	1	7	4					3	2				
600	-21	-8	-28	-16	-21	-12					-5	-4	-1	-1		

$\text{Spin}(7) (B_3)$

λ	0		A_1		B_1		$A_1 \oplus B_1$		A_2		B_2		D_2		$D_2 \oplus B_1$		D_3	
000	231	48	1001	312	693	216	1155	540	246	160	126	108	330	144	470	312	7	8
002	-27	6	-99	36	-63	24	231	144	-25	5	-60	-36	21	33	214	144	-1	-1
004	-24		1	24	-162	-48	-60	-36	-1	4					20	12		
006	15	6	20	12					1	1								
010	-63	-8	-147	-40	-336	-52	-616	-120	51		-18	-24	-63	-14	-327	-69	3	2
012	-3	-4	-39	-22	56	-8		-12	-6	-10	10	3	14	4	-45	-12		
014	-27	-4	-45	-12	-9	-2			-3	-2								
020	-36	-8	-108	-36	-60	-36	-12	-34	12	8	3	6	-42	-8	79	36		
022	15		36	4	-7	-8	-21	-16	3	1			-5	-3				
030	-15	-4	-23	-8	-24	-18	24		-1				15	6	1	1		
032	3	2	1	1														
040	-9	-4	-3	-2	-3	-2	-1	-1										
100	-21	-8	-175	-60	126	-16		-72	-82	-40	60	14	-126	-40	-120	-72	-5	-4
102	57	4	123	12	229	34	189	48	-3	-2	-4		-28	-12	-27	-14		
104	18	2	24	5	42	8	35	12	3	2								
110	36	4	198	30	-84		-168	-32	35	20	15	11	84	8	-10	-32		
112	18	2	27	2	48	17	-12	8	1	2			6	2	-5	-3		
114	-6	-2	-5	-3														
120	-9	4	-21	16	-12	12	56	32	-9	-4	-3	-1	-18	-4	6			
122	12	2	12	4	4	1	3	2										
130	-6		-11	-2	6	4		2										
200	-42		-210	-16	28	32	84	88	-90	-48	28	32	13	16	144	112		
202	-48	-4	-118	-20	-84	-8	-84	-32	-4	-3	-6	-4	-12	-7	22	16		
204	9	4	12	10	3	2	-5	-3										
210	27	2	117	20	-45	-15	3	-20	6			-4	36	16	-30	-11		
212	-15	-2	-17	-5	-21	-4							3	2				
220			-9	-4	18	8	-12	-10					-9	-4				
222	-1	-1																
230	3	2			1	1												
300	-18	-8	-90	-52	42		126	72	14	8	-3	-9	-63	-36	-9			
302	-9	-4	-9	-8	-24	-18	24	12					-6	-6	1	1		
304	3	2	1	1														
310	36	4	66	16	48	4	-14	-2			3	2	18	8	-3	-2		
312	-6	-2	-3	-2	-2	-1												
320	3		4	2	-3	-4												
400	-9		-45	-16	39	32	-15	-8			2	-9	-8		5	4		
402	6	2			14	8							-1	-1				
410	3	2	7	4	-6	1							3	2				
500	-21	-8	-21	-12	-28	-16					-1	-1	-5	-4				

Spin(8) (D_4)

λ	0		A_1		$A_1 \oplus A_1$		A_2		D_2		$A_1 \oplus D_2$		A_3		D_3		$D_2 \oplus D_2$	
0000	1617	192	15092	2688	6468	1728	8547	3360	6468	1728	12936	5184	381	352	381	352	3430	1944
0002	-189	24	-1512	312	1134	438	-1749	-48		288	3234	1584	111	105	-135	-96	1470	864
0004	27	24	162	240	483	240	-30	132	-462	-144	-154	-144	3	12			105	54
0006	105	24	350	120	105	54	50	32					1	1				
0020	-189	24	-1512	312		288	-1749	-48		288	3234	1584	-135	-96	-135	-96	1470	864
0022	-324	-12	-2052	-132	-420	-48	-108	40	-996	-183	-336	-120	-10	-8	11	4	245	144
0024	54	12	216	84	105	72	9	12	-84	-24	-70	-36						
0040	27	24	162	240	-462	-144	-30	132	-462	-144	-154	-144					105	54
0042	54	12	216	84	-30	-9	9	12	-84	-24	-70	-36						
0044	-15	-6	-20	-12			-1	-1										
0060	105	24	350	120			50	32										
0100	-588	-32	-4900	-432	-2940	-296	-462	-408	-2940	-296	-8526	-988	38	-24	38	-24	-3136	-432
0102	27	-8	180	-92	-420	-68	-207	-136	630	12	84	-6	-9	-18	15	4	-441	-72
0104	-162	-8	-702	-60	-273	-44	-108	-24	42	4	84	14	-3	-2				
0120	27	-8	180	-92	630	12	-207	-136	630	12	84	-6	15	4	15	4	-441	-72
0122	-108	-4	-504	-34	-90	-24	-90	-24	78	-4	186	30						
0124	27	4	54	14			3	2										
0140	-162	-8	-702	-60	42	4	-108	-24	42	4	84	14						
0142	27	4	54	14			3	2										
0200	-216	-24	-1512	-288	-420	-64	18	-312	-420	-64	490	256	30	24	30	24	1029	364
0202	54	-4	270	-36	225	47	132	-32	-180	-64	-384	-128	3	1			21	16
0204	-6	4	-14	16	21	16	-8	-4										
0220	54	-4	270	-36	-180	-64	132	-32	-180	-64	-384	-128					21	16
0222	-54	-4	-144	-20	9	8	-9	-4	-6	-1	-6	-4						
0240	-6	4	-14	16			-8	-4										
0300	-87	-24	-464	-228	-30	-90	177	152	-30	-90	570	27	-1		-1		-49	
0302	27		81		-57	-18	24	8	-30	-18	-4	-9						
0320	27		81		-30	-18	24	8	-30	-18	-4	-9						
0322	-3	-2	-1	-1														
0400	-51	-8	-170	-48	75	8	-48	-12	75	8	6	-8					1	1
0402	9	4	6	4	1	1												
0420	9	4	6	4														
0500	-27	-8	-27	-12	-3	-2			-3	-2	-1	-1						
1011	606	10	4444	125	798	4	1760	244	798	4	-294	-248	-2	12	-2	12	-392	-192
1013	162		864		75	-26	189	18	300	46	150	49	6	4			-35	-12
1015	-45	-6	-120	-27	-35	-12	-6	-4										
1031	162		864		300	46	189	18	300	46	150	49					-35	-12
1033	-27	-4	-81	-22	-15	-6	-6	-6	12	2	10	3						
1051	-45	-6	-120	-27			-6	-4										
1111	96	14	560	147	120	60	-320	24	120	60	60	63	-8	-2	-8	-2		-32
1113	54		189		36	-10	24	8	36	8	18	11						
1131	54		189		36	8	24	8	36	8	18	11						
1133	6	2	5	3														
1211	27	6	108	45	21	20	-45	-6	21	20	16	25					-3	-2
1213	-12	-2	-16	-5	-3	-2												
1231	-12	-2	-16	-5														
1311	24	2	44	7	6	2			6	2	3	2						
2000	-189	24	-1512	312		288	-1749	-48	1134	438	3234	1584	-135	-96	111	105	1470	864
2002	-324	-12	-2052	-132	-420	-48	-108	40	-420	-48	-336	-120	-10	-8	-10	-8	245	144
2004	54	12	216	84	105	72	9	12	-30	-9	-70	-36						
2020	-324	-12	-2052	-132	-996	-183	-108	40	-420	-48	-336	-120	11	4	-10	-8	245	144
2022	-69	-2	-299	-16	-111	-8	14	20	-111	-8	-186	-80					5	3
2024	-9	-4	-15	-12	5	3												
2040	54	12	216	84	-84	-24	9	12	-30	-9	-70	-36						
2042	-9	-4	-15	-12														
2100	27	-8	180	-92	630	12	-207	-136	-420	-68	84	-6	15	4	-9	-18	-441	-72
2102	-108	-4	-504	-34	-90	-24	-90	-24	-90	-24	186	30						
2104	27	4	54	14			3	2										
2120	-108	-4	-504	-34	78	-4	-90	-24	-90	-24	186	30						
2122	33	6	77	27	-6	-2			-6	-2	-3	-1						
2140	27	4	54	14			3	2										
2200	54	-4	270	-36	-180	-64	132	-32	225	47	-384	-128			3	1	21	16
2202	-54	-4	-144	-20	9	8	-9	-4	9	8	-6	-4						
2220	-54	-4	-144	-20	-6	-1	-9	-4	9	8	-6	-4						
2222	1	1																
2300	27		81		-30	-18	24	8	-57	-18	-4	-9						
2302	-3	-2	-1	-1														
2320	-3	-2	-1	-1														
2400	9	4	6	4					1	1								
3011	162		864		300	46	189	18	75	-26	150	49			6	4	-35	-12
3013	-27	-4	-81	-22	-15	-6	-6	-6	-15	-6	10	3						
3031	-27	-4	-81	-22	12	2	-6	-6	-15	-6	10	3						
3033	-3	-2	-1	-1														
3111	54		189		36	8	24	8	36	-10	18	11						
3113	6	2	5	3														
3131	6	2	5	3														
3211	-12	-2	-16	-5					-3	-2								
4000	27	24	162	240	-462	-144	-30	132	483	240	-154	-144			3	12	105	54
4002	54	12	216	84	-30	-9	9	12	105	72	-70	-36						
4004	-15	-6	-20	-12			-1	-1										
4020	54	12	216	84	-84	-24	9	12	105	72	-70	-36						
4022	-9	-4	-15	-12					5	3								

Spin(8) (D_4) (continued)

λ	0		A_1		$A_1 \oplus A_1$		A_2		D_2		$A_1 \oplus D_2$		A_3	D_3		$D_2 \oplus D_2$	
4040	-15	-6	-20	-12			-1	-1									
4100	-162	-8	-702	-60	42	4	-108	-24	-273	-44	84	14		-3	-2		
4102	27	4	54	14			3	2									
4120	27	4	54	14			3	2									
4200	-6	4	-14	16			-8	-4	21	16							
5011	-45	-6	-120	-27			-6	-4	-35	-12							
6000	105	24	350	120			50	32	105	54				1	1		

References

- [1] R. Abraham, J.E. Marsden: *Foundations of Mechanics*. Benjamin/Cummings 1978
- [2] A. Borel, J. de Siebenthal: Les sous-groupes fermés de rang maximum des groupes de Lie clos. *Comment. Math. Helv.* **23** (1949) 200–221
- [3] N. Bourbaki: *Lie groups and Lie algebras*. Chapters 7–9. Springer 2005
- [4] T. Bröcker, T. tom Dieck: *Representations of compact Lie groups*. Springer 1995
- [5] R. Carter: Conjugacy classes in the Weyl group. *Compositio Math.* **25** (1972) 1–59
- [6] S. Charzyński, J. Kijowski, G. Rudolph, M. Schmidt: On the stratified classical configuration space of lattice QCD. *J. Geom. Phys.* **55** (2005) 137–178
- [7] S. Charzyński, G. Rudolph, M. Schmidt: On the topological structure of the stratified classical configuration space of lattice QCD. *J. Geom. Phys.* **58** (2008) 1607–1623
- [8] J.M. Douglass, G. Pfeiffer, G. Röhrle: On reflection subgroups of finite Coxeter groups. Oberwolfach preprints, Mathematisches Forschungsinstitut Oberwolfach, 2011. Preprint arXiv:1101.5893v3
- [9] M.J. Dyer, G.I. Lehrer: Reflection subgroups of finite and affine Weyl groups. *Trans. Amer. Math. Soc.* **363** (2011) 5971–6005
- [10] E.B. Dynkin: Semisimple subalgebras of semisimple Lie algebras (in Russian). *Matematičeskij Sbornik* **30** (1952) 349–462. English Translation in *Am. Math. Soc. Transl., Ser. 2*, **6** (1957) 111–244
- [11] R. Feger, T.W. Kephart: LieART—a Mathematica application for Lie algebras and representation theory. Preprint arXiv:1206.6379. Package hosted at <http://lieart.hepforge.org/>
- [12] A. Felikson, P. Tumarkin: Reflection subgroups of Euclidean reflection groups. *Sb. Math.* **196** (2005) 1349–1369. Preprint arXiv:math/0402403v2
- [13] E. Fischer, G. Rudolph, M. Schmidt: A lattice gauge model of singular Marsden-Weinstein reduction. Part I. Kinematics. *J. Geom. Phys.* **57** (2007) 1193–1213

- [14] M. Geck, G. Hiss, F. Lübeck, G. Malle, G. Pfeiffer: CHEVIE – A system for computing and processing generic character tables for finite groups of Lie type, Weyl groups and Hecke algebras. *Appl. Algebra Engrg. Comm. Comput.* **7** (1996) 175–210
- [15] H. Grundling, G. Rudolph: QCD on an infinite lattice. *Commun. Math. Phys.* (2013) DOI 10.1007/s00220-013-1674-5, arXiv:1108.2129v2 [math-ph]
- [16] W.Y. Hsiang: *Lectures on Lie groups*. World Scientific 2000
- [17] B.C. Hall: The Segal-Bargmann coherent state transform for compact Lie groups. *J. Funct. Anal.* **122** (1994) 103–151
- [18] B.C. Hall: Geometric quantization and the generalized Segal-Bargmann transform for Lie groups of compact type. *Commun. Math. Phys.* **226** (2002) 233–268
- [19] J. Huebschmann: Kähler spaces, nilpotent orbits, and singular reduction. *Memoirs of the AMS* **172** (814), Amer. Math. Soc., Providence R.I., , 2004. Preprint [math.DG/0104213](#)
- [20] J. Huebschmann: Singular Poisson-Kähler geometry of certain adjoint quotients, in: *Proceedings, The mathematical legacy of C. Ehresmann*, Bedlewo, 2005, Banach Center Publications **76** (2007) 325–347. Preprint [math.SG/0610614](#)
- [21] J. Huebschmann: Kirillov’s character formula, the holomorphic Peter-Weyl theorem, and the Blattner-Kostant-Sternberg pairing. *J. Geom. Phys.* **58** (2008) 833–848. Preprint [math.DG/0610613](#)
- [22] J. Huebschmann, G. Rudolph, M. Schmidt: A lattice gauge model for quantum mechanics on a stratified space. *Commun. Math. Phys.* **286** (2009) 459–494
- [23] P.D. Jarvis, J. Kijowski, G. Rudolph: On the structure of the observable algebra of QCD on the lattice. *J. Phys. A* **38** (2005) 5359–5377
- [24] J. Kijowski, G. Rudolph: On the Gauss law and global charge for quantum chromodynamics. *J. Math. Phys.* **43** (2002) 1796–1808
- [25] J. Kijowski, G. Rudolph: Charge superselection sectors for qcd on the lattice. *J. Math. Phys.* **46** (2005) 032303
- [26] R.V. Moody, J. Patera: Fast Recursion Formula for Weight Multiplicities. *Bull. Amer. Math. Soc.* **7** (1) (1982) 237–242.
- [27] J.-P. Ortega, T.S. Ratiu: *Momentum Maps and Hamiltonian Reduction*. Progress in Mathematics, Vol. 222, Birkhuser 2004
- [28] M.J. Pflaum: *Analytic and Geometric Study of Stratified Spaces*. Lecture Notes in Mathematics 1768, Springer 2001

- [29] G. Rudolph, M. Schmidt, I.P. Volobuev: Classification of Gauge Orbit Types for SUn-Gauge Theories. *J. Math. Phys. Anal. Geom.* **5** (2002) 201–241
- [30] G. Rudolph, M. Schmidt: On the algebra of quantum observables for a certain gauge model. *J. Math. Phys.* **50** (2009) 052102
- [31] M. Schmidt: Classification and partial ordering of reductive Howe dual pairs of classical Lie group. *J. Geom. Phys.* **29** (1999) 283–318
- [32] M. Schönert et al.: GAP – Groups, Algorithms, and Programming – version 3 release 4 patchlevel 4. Lehrstuhl für Mathematik, RWTH Aachen, 1997
- [33] B. Simon: *Representations of Finite and Compact Groups*. Graduate Studies in Mathematics, Vol. 10, American Mathematical Society 1996
- [34] R. Sjamaar, E. Lerman: Stratified symplectic spaces and reduction. *Ann. of Math.* **134** (1991) 375–422
- [35] Wolfram Research, Inc.: Mathematica, Version 8.0. Champaign, IL, 2010